Problem 12332
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Proposed by F. Holland (Ireland).

Prove

$$\int_0^{\infty} \frac{\tanh^2(x)}{x^2}\, dx = \frac{14\zeta(3)}{\pi^2}.$$

Solution proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.

Solution 1. By the residue theorem, for any positive integer k,

$$\int_{\gamma_k} \frac{\tanh^2(z)}{z^2}\, dz = 2\pi i \sum_{j=1}^{k} \operatorname{Res}\left(\frac{\tanh^2(z)}{z^2}, z_j\right)$$

where γ_k is the boundary of the rectangle $[-k,k]\times[0,i\pi k]$ oriented counter-clockwise and $z_j = i\pi(j-1/2)$ for $j\in\mathbb{Z}$ are the poles of the integrand function.
We note that along the two vertical sides of γ_k,

$$|\tanh(\pm k + iy)| = \left|\frac{e^{\pm k+iy} - e^{\mp k-iy}}{e^{\pm k+iy} + e^{\mp k-iy}}\right| \le \frac{e^k + e^{-k}}{e^k - e^{-k}} \le \frac{1}{\tanh(k)} \le \frac{1}{\tanh(1)} < 2,$$

whereas, along the horizontal upper side, $|\tanh(x+i\pi k)| = |\tanh(x)| < 1$. Hence, as $k\to+\infty$,

$$\left|\int_{\gamma_k\setminus[-k,k]} \frac{\tanh^2(z)}{z^2}\, dz\right| \le 2\int_0^{\pi k} \frac{4}{k^2+y^2}\, dy + \int_{-k}^{k} \frac{1}{x^2+\pi^2k^2}\, dx \le \frac{8\pi k}{k^2} + \frac{2k}{\pi^2k^2} = \frac{8\pi^3+2}{\pi^2 k} \to 0.$$

Moreover, the following series expansion at $z=z_j$ holds: as $w = z - z_j \to 0$,

$$\tanh(z) = \tanh(z_j + w) = \frac{1}{\tanh(w)} = \frac{1}{w + O(w^3)} = \frac{1}{w} + O(w).$$

Therefore

$$\frac{\tanh^2(z)}{z^2} = \frac{\left(\frac{1}{w} + O(w)\right)^2}{(z_j+w)^2} = \frac{1}{z_j^2}\left(\frac{1}{w^2} + O(1)\right)\left(1 - \frac{2w}{z_j} + O(w^2)\right) = \frac{1}{z_j^2 w^2} - \frac{2}{z_j^3 w} + O(1)$$

which implies that

$$\operatorname{Res}\left(\frac{\tanh^2(z)}{z^2}, z_j\right) = -\frac{2}{z_j^3} = -\frac{2i}{\pi^3(j-1/2)^3} = -\frac{16i}{\pi^3(2j-1)^3}.$$

Finally,

$$\begin{aligned}\int_0^{\infty} \frac{\tanh^2(x)}{x^2}\, dx &= \frac{1}{2}\lim_{k\to\infty}\int_{-k}^{k} \frac{\tanh^2(x)}{x^2}\, dx = \frac{1}{2}\lim_{k\to\infty}\int_{\gamma_k} \frac{\tanh^2(z)}{z^2}\, dz \\ &= \pi i \sum_{j=1}^{\infty} \operatorname{Res}\left(\frac{\tanh^2(z)}{z^2}, z_j\right) = \pi i \sum_{j=1}^{\infty} \frac{-16i}{\pi^3(2j-1)^3} = \frac{16}{\pi^2}\cdot\frac{7\zeta(3)}{8} = \frac{14\zeta(3)}{\pi^2}.\end{aligned}$$

□

Solution 2. We have that

$$\begin{aligned}\int_0^{\infty} \frac{\tanh^2(x)}{x^2}\,dx &= \frac{1}{2}\int_{-\infty}^{+\infty} \frac{\tanh^2(x)}{x^2}\,dx = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{(1-e^{-2x})^2}{(1+e^{-2x})^2}\frac{1}{x^2}\,dx \\ &= \frac{1}{2}\int_0^{+\infty} \frac{(1-s)^2}{(1+s)^2}\frac{4}{\log^2(s)}\frac{ds}{2s} \qquad [s=e^{-2x}] \\ &= \int_0^{+\infty} \frac{(1-s)^2}{(1+s)^2} D\left(-\frac{1}{\log(s)}\right)\,ds \\ &= \left[-\frac{(1-s)^2}{(1+s)^2\log(s)}\right]_{0^+}^{+\infty} + \int_0^{+\infty} D\left(\frac{(1-s)^2}{(1+s)^2}\right)\frac{ds}{\log(s)} \\ &= 4\int_0^{+\infty} \frac{s-1}{(1+s)^3\log(s)}\,ds \\ &= 4I(1)\end{aligned}$$

where

$$I(u) = \int_0^{+\infty} \frac{s^u-1}{(1+s)^3\log(s)}\,ds.$$

By the Euler's reflection formula, for $u \in (0,1)$,

$$\begin{aligned}I'(u) &= \int_0^{+\infty} \frac{s^u}{(1+s)^3}\,ds \overset{w=\frac{1}{1+s}}{=} \int_0^1 w^{1-u}(1-w)^u\,dw = B(2-u,u+1) \\ &= \frac{\Gamma(2-u)\Gamma(u+1)}{\Gamma(3)} = \frac{u(1-u)}{2}\Gamma(u)\Gamma(1-u) = \frac{\pi u(1-u)}{2\sin(\pi u)}.\end{aligned}$$

Therefore

$$\begin{aligned}I(1) &= \int_0^1 I'(u)\,du = \frac{\pi}{2}\int_0^1 \frac{u(1-u)}{\sin(\pi u)}\,du = \pi i\int_0^1 \frac{u(1-u)}{e^{i\pi u}-e^{-i\pi u}}\,du \\ &= \pi i\sum_{k=0}^{\infty}\int_0^1 u(1-u)e^{-i(2k+1)\pi u}\,du = \frac{4}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3} = \frac{4}{\pi^2}\zeta(3)\left(1-\frac{1}{8}\right) = \frac{7\zeta(3)}{2\pi^2}.\end{aligned}$$

Finally,

$$\int_0^{\infty} \frac{\tanh^2(x)}{x^2}\,dx = 4I(1) = 4\frac{7\zeta(3)}{2\pi^2} = \frac{14\zeta(3)}{\pi^2}.$$

□