Problem 12331
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Proposed by WeChat Group on Matrix Analysis (USA).

Let A and B be complex $m \times n$ matrices, and let C be a complex $n \times m$ matrix. Prove that if there are nonzero scalars x and y such that $ACB = xA + yB$, then $ACB = BCA$.

Solution proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.

Solution. We note that AC and BC are $n \times n$ matrices such that

$$(AC - yI)(BC - xI) = ACBC - (xA + yB)C + xyI = xyI.$$

Since $xy \neq 0$, it follows that $(AC - yI)$ and $(BC - xI)$ are invertible. Therefore

$$BC = xI + xy(AC - yI)^{-1}. \tag{1}$$

In a similar way, CA and CB are $m \times m$ matrices, and from

$$(CA - yI)(CB - xI) = CACB - C(xA + yB) + xyI = xyI$$

we find

$$CB = xI + xy(CA - yI)^{-1}. \tag{2}$$

By (1) and (2),

$$BCA = (BC)A = (xI + xy(AC - yI)^{-1})A = xA + xy(AC - yI)^{-1}A$$
$$ACB = A(CB) = A(xI + xy(CA - yI)^{-1}) = xA + xyA(CA - yI)^{-1}$$

Hence $ACB = BCA$ if and only if $A(CA - yI)^{-1} = (AC - yI)^{-1}A$, or

$$(AC - yI)A = ACA - yA = A(CA - yI)$$

which is trivially true. □