Problem 12328
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An integer binary quadratic form is a function $f : \mathbb{Z}^2 \to \mathbb{Z}$ *defined by* $f(m,n) = am^2 + bmn + cn^2$ *for some* $a, b, c \in \mathbb{Z}$*. The value set* $V(f)$ *of such a form is defined to be* $\{f(m,n) : (m,n) \in \mathbb{Z}^2\}$*.*
(a) Prove that if $f_1(m,n) = m^2 - mn - 3n^2$ *and* $f_2(m,n) = m^2 - 13n^2$*, then* $V(f_1) = V(f_2)$*.*
(b) Prove that if $f_1(m,n) = m^2 - mn - 4n^2$ *and* $f_2(m,n) = m^2 - 17n^2$*, then* $V(f_1) \supseteq V(f_2)$ *but* $V(f_1) \neq V(f_2)$*.*

Solution proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.

Solution. We consider the following linear transformations and their inverses:

$$\begin{cases} T_1(x,y) = (x+y, 2y) \\ T_1^{-1}(x,y) = (x - \frac{y}{2}, \frac{y}{2}) \end{cases}, \begin{cases} T_2(x,y) = (4x - 14y, -3x + 11y) \\ T_2^{-1}(x,y) = (\frac{11x}{2} + 7y, \frac{3x}{2} + 2y) \end{cases}, \begin{cases} T_3(x,y) = (7x - 25y, 3x - 11y) \\ T_3^{-1}(x,y) = (\frac{11x - 25x}{2}, \frac{3x - 7y}{2}) \end{cases}.$$

(a) It is straightforward to verify that $f_1(T_k(x,y)) = f_2(x,y)$ for any $x, y \in \mathbb{R}$ and for $k = 1, 2, 3$.
If $N \in V(f_2)$ then $f_2(m,n) = N$ for some $(m,n) \in \mathbb{Z}^2$ and therefore $f_1(T_1(m,n)) = N$ with $T_1(m,n) \in \mathbb{Z}^2$ which implies that $N \in V(f_1)$.
If $N \in V(f_1)$ then $f_1(m,n) = N$ for some $(m,n) \in \mathbb{Z}^2$. We have three cases:
i) If m is even then $T_2^{-1}(m,n) \in \mathbb{Z}^2$ and $f_2(T_2^{-1}(m,n)) = N$ which implies that $N \in V(f_2)$.
ii) If n is even then $T_1^{-1}(m,n) \in \mathbb{Z}^2$ and $f_2(T_1^{-1}(m,n)) = N$ which implies that $N \in V(f_2)$.
iii) If m and n are both odd then $T_3^{-1}(m,n) \in \mathbb{Z}^2$ and $f_2(T_3^{-1}(m,n)) = N$ which implies that $N \in V(f_2)$.
Thus, we may conclude that $V(f_1) = V(f_2)$.

(b) It is straightforward to verify that $f_1(T_1(x,y)) = f_2(x,y)$ for any $x, y \in \mathbb{R}$.
If $N \in V(f_2)$ then $f_2(m,n) = N$ for some $(m,n) \in \mathbb{Z}^2$ and therefore $f_1(T_1(m,n)) = N$ with $T_1(m,n) \in \mathbb{Z}^2$ which implies that $N \in V(f_1)$.
Notice that $f_1(3,1) = 3^2 - 3 - 4 = 2 \in V(f_1)$. On the other hand, $2 \notin V(f_2)$ because if $m^2 - 17n^2 = 2$ for some $(m,n) \in \mathbb{Z}^2$ then

$$m^2 - n^2 \equiv 2 \pmod 4$$

which is impossible since a square modulo 4 is congruent to 0 or 1 and therefore $m^2 - n^2 \in \{0, 1, 3\}$ (mod 4). Thus, we may conclude that $V(f_1) \supsetneq V(f_2)$. □