Problem 12327
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Proposed by M. Merca (Romania).

For $n \geq 0$, prove

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} q^k = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} q^{k(k-1)+(n-k)^2-n(n-1)/2}.$$

Solution proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.

Solution. We show by induction that both sides are equal to $P(n) = \prod_{k=1}^{n}(1+q^k)$.
i) For $n = 0$, the left-hand side is equal to $1 = P(0)$. Let $n \geq 1$. Since

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k \end{bmatrix}_q = q^{n-k} \begin{bmatrix} n-1 \\ n-k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k \end{bmatrix}_q,$$

by the induction hypothesis,

$$\begin{aligned}
\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} q^k &= \sum_{k=1}^{n} q^{2(n-k)} \begin{bmatrix} n-1 \\ n-k \end{bmatrix}_{q^2} q^k + \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{q^2} q^k \\
&= q^n \sum_{k=1}^{n} \begin{bmatrix} n-1 \\ n-k \end{bmatrix}_{q^2} q^{n-k} + \prod_{k=1}^{n-1}(1+q^k) \\
&= q^n \prod_{k=1}^{n-1}(1+q^k) + \prod_{k=1}^{n-1}(1+q^k) = \prod_{k=1}^{n}(1+q^k).
\end{aligned}$$

ii) For $n = 0$, the right-hand side is equal to $1 = P(0)$. Let $n \geq 1$. Since

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ n-k \end{bmatrix}_q + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q,$$

by the induction hypothesis,

$$\begin{aligned}
\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} q^k &= \sum_{k=1}^{n} \begin{bmatrix} n-1 \\ n-k \end{bmatrix}_q q^{a(n,k)} + \sum_{k=0}^{n-1} q^{2k} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q q^{a(n,k)} \\
&= \sum_{k=1}^{n} \begin{bmatrix} n-1 \\ n-k \end{bmatrix}_q q^{a(n-1,n-k)} + q^n \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q q^{a(n-1,k)} \\
&= \prod_{k=1}^{n-1}(1+q^k) + q^n \prod_{k=1}^{n-1}(1+q^k) = \prod_{k=1}^{n}(1+q^k)
\end{aligned}$$

where $a(n,k) = k(k-1) + (n-k)^2 - n(n-1)/2$ and it is easy to verify that

$$a(n,k) = a(n-1,n-k) \qquad \text{and} \qquad a(n,k) + 2k = a(n-1,k) + n.$$

$\square$