Problem 12326
(American Mathematical Monthly, Vol.129, May 2022)

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Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that, for every fixed $y \in \mathbb{R}$, $f(x+y) - f(x)$ is a polynomial in x. Prove that f is a polynomial function.

Solution proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.

Solution. Given $y \neq 0$ and $z \neq 0$,

$P_y(x) := f(x+y)-f(x) = a_n x^n + \cdots + a_1 x + a_0$ and $P_z(x) := f(x+z)-f(x) = b_m x^m + \cdots + b_1 x + b_0$

are polynomials in x of degree n, m and leading coefficients, a_n, b_m respectively.
Since $f(x+y+z) - f(y) - f(z) + f(x)$ can be written in two ways:

$$f(x+y+z) - f(y) - f(z) + f(x) = P_y(x+z) - P_y(x) = a_n nzx^{n-1} + o(x^{n-1}),$$
$$f(x+y+z) - f(y) - f(z) + f(x) = P_z(x+y) - P_z(x) = b_m myx^{m-1} + o(x^{m-1}),$$

it follows that $m = n$ and $a_n/y = b_m/z := c$, which means that the polynomials $P_y(x)$ have all the same degree $n \geq 0$ with leading coefficient cy (the same n and c for all $y \neq 0$).
If $c > 0$ then let us consider the continuous function given by

$$f_1(x) := f(x) - \frac{cx^{n+1}}{n+1}.$$

f_1 has the same property of f: $P_y(x) := f_1(x+y) - f_1(x)$ is a polynomial in x for all $y \neq 0$, but the common degree of all the polynomials $P_y(x)$ is now less than n,

$$f_1(x+y) - f_1(x) = f(x+y) - f(x) - \frac{c((x+y)^{n+1} - x^{n+1})}{n+1} = cyx^n - cyx^n + o(x^n) = o(x^n).$$

Therefore, in a finite number of steps, say N, we get a continuous function f_N where $P_y(x) := f_N(x+y) - f_N(x)$ is identically 0 for any y, which implies that f_N is a constant polynomial. Thus, going backwards from f_N to f, we may conclude that also f is a polynomial. $\square$