Problem 12324
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Proposed by A. Stadler (Switzerland).

Let a and b be positive real numbers. Prove

$$\int_0^\infty \frac{dx}{\sqrt{ax^4+2(2b-a)x^2+a}} = \int_0^\infty \frac{dx}{\sqrt{bx^4+2(2a-b)x^2+b}}.$$

Solution proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.

Solution. We have that

$$\begin{aligned}
\int_0^\infty \frac{dx}{\sqrt{ax^4+2(2b-a)x^2+a}} &= \int_0^\infty \frac{1}{\sqrt{a+(b-a)(\frac{2x}{1+x^2})^2}}\cdot\frac{dx}{1+x^2} \\
&\overset{x=\tan(t)}{=} \int_0^{\pi/2} \frac{1}{\sqrt{a+(b-a)(2\sin(t)\cos(t))^2}}\,dt \\
&= \int_0^{\pi/2} \frac{dt}{\sqrt{a+(b-a)\sin^2(2t)}} = \int_0^{\pi/2} \frac{dt}{\sqrt{a\cos^2(2t)+b\sin^2(2t)}} \\
&\overset{s=2t}{=} \frac{1}{2}\int_0^{\pi} \frac{ds}{\sqrt{a\cos^2(s)+b\sin^2(s)}} = \int_0^{\pi/2} \frac{ds}{\sqrt{a\cos^2(s)+b\sin^2(s)}} \\
&= \int_0^{\pi/2} \frac{1}{\sqrt{a+b\tan^2(s)}}\cdot\frac{ds}{\cos(s)} \\
&\overset{u=\sqrt{b}\tan(s)}{=} \int_0^\infty \frac{du}{\sqrt{a+u^2}\sqrt{b+u^2}}
\end{aligned}$$

which is invariant under the exchange of a and b.
Therefore we may conclude that the two given integrals are equal. □