Problem 12323
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(a) Find integers c_0, c_1, and c_2 such that

$$\sum_{k=0}^{\infty} \frac{k^{11}}{(k!)^3} = \sum_{k=0}^{\infty} \frac{c_0 + c_1 k + c_2 k^2}{(k!)^3}.$$

(b) Prove that for any integers n and b with $1 \leq b \leq n$, there are integers $c_0, \ldots, c_{b-1}$ such that

$$\sum_{k=0}^{\infty} \frac{k^n}{(k!)^b} = \sum_{k=0}^{\infty} \frac{\sum_{m=0}^{b-1} c_m k^m}{(k!)^b}.$$

(c) Prove that the integers $c_0, \ldots, c_{b-1}$ from part (b) are unique.

Solution proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.

Solution. We start by proving (b), then (a) and finally (c).

(b) Let $S(n,b) = \sum_{k=0}^{\infty} \frac{k^n}{(k!)^b}$ then the following recurrence holds for $1 \leq b \leq n$,

$$S(n,b) = \sum_{k=1}^{\infty} \frac{k^{n-b}}{((k-1)!)^b} = \sum_{k=0}^{\infty} \frac{(k+1)^{n-b}}{(k!)^b} = \sum_{k=0}^{\infty} \frac{1}{(k!)^b} \sum_{m=0}^{n-b} \binom{n-b}{m} k^m = \sum_{m=0}^{n-b} \binom{n-b}{m} S(m,b).$$

Hence, by induction on $n \geq b$, it is easy to verify that $S(n,b)$ can be written as an integer linear combination of $S(m,b)$ for $m = 0, \ldots, b-1$:

$$S(n,b) = \sum_{m=0}^{b-1} c_m S(m,b).$$

Note that for $n \geq b = 1$, c_0 is the n-th Bell number: $1, 2, 5, 15, 52, 203, 877, 4140, \ldots$

(a) As an application of (b), we find integers c_0, c_1, and c_2 for $n = 11$ and $b = 3$:

$$\begin{aligned}
S(3,3) &= S(0,3)\\
S(4,3) &= S(0,3) + S(1,3)\\
S(5,3) &= S(0,3) + 2S(1,3) + S(2,3)\\
S(6,3) &= S(0,3) + 3S(1,3) + 3S(2,3) + S(3,3) = 2S(0,3) + 3S(1,3) + 3S(2,3)\\
S(7,3) &= S(0,3) + 4S(1,3) + 6S(2,3) + 4S(3,3) + S(4,3) = 6S(0,3) + 5S(1,3) + 6S(2,3)\\
S(8,3) &= S(0,3) + 5S(1,3) + 10S(2,3) + 10S(3,3) + 5S(4,3) + S(5,3) = 17S(0,3) + 12S(1,3) + 11S(2,3)\\
S(9,3) &= 44S(0,3) + 36S(1,3) + 24S(2,3)\\
S(10,3) &= 112S(0,3) + 110S(1,3) + 69S(2,3)\\
S(11,3) &= 304S(0,3) + 326S(1,3) + 227S(2,3).
\end{aligned}$$

Therefore $c_0 = 304$, $c_1 = 326$, $c_2 = 227$.

(c) Assume that there are $n \geq b \geq 1$ and integers $c_0, \ldots, c_{b-1}$, $c'_0, \ldots, c'_{b-1}$ such that

$$\sum_{k=0}^{\infty} \frac{k^n}{(k!)^b} = \sum_{k=0}^{\infty} \frac{\sum_{m=0}^{b-1} c_m k^m}{(k!)^b} = \sum_{k=0}^{\infty} \frac{\sum_{m=0}^{b-1} c'_m k^m}{(k!)^b}.$$

Then, by taking the difference, we have that

$$\sum_{k=0}^{\infty} \frac{P(k)}{(k!)^b} = 0$$

where $P(x) = \sum_{m=0}^{b-1}(c_m - c'_m)x^m$ is a polynomial with integer coefficients of degree d less than b. Thus the required uniqueness property holds as soon as we show that P is identically zero.
If it is not, then we may assume without loss of generality that the leading coefficient of P is positive. Let N be a sufficiently large positive integer such that $0 < P(k)/k^b < 1/2$ for all $k > N$. Then we have

$$0 = (N!)^b \sum_{k=0}^{N} \frac{P(k)}{(k!)^b} + (N!)^b \sum_{k=N+1}^{\infty} \frac{P(k)}{(k!)^b} = A + B$$

which is a contradiction because A is an integer,

$$A := (N!)^b \sum_{k=0}^{N} \frac{P(k)}{(k!)^b} = \sum_{k=0}^{N} P(k) \left(\frac{N!}{k!}\right)^b \in \mathbb{Z},$$

and $B \in (0, 1)$,

$$\begin{aligned} 0 < B : = (N!)^b \sum_{k=N+1}^{\infty} \frac{P(k)}{(k!)^b} &= \sum_{k=N+1}^{\infty} \frac{P(k)}{k^b} \cdot \left(\frac{N!}{(k-1)!}\right)^b \\ &< \frac{1}{2}\left(1 + \frac{1}{(N+1)^b} + \frac{1}{((N+2)(N+1))^b} + \dots\right) < \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \dots\right) = 1. \end{aligned}$$

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