Problem 12322
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Given real numbers $x_1, \dots, x_{2n}$, *let* A_n *be the skew-symmetric* $2n$*-by-*$2n$ *matrix with entries* $a_{i,j} = (x_i - x_j)^2$ *for* $1 \le i < j \le 2n$. *Prove*

$$\det(A_n) = 4^{n-1} \left((x_1 - x_2)(x_2 - x_3) \cdots (x_{2n-1} - x_{2n})(x_{2n} - x_1)\right)^2.$$

Solution proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.

Solution. It is known that, for any skew-symmetric matrix M, $\det(M) = \mathrm{pf}^2(M)$ where $\mathrm{pf}(M)$ is the Pfaffian of the matrix M. Hence it suffices to prove that

$$\mathrm{pf}(A_n) = (-1)^n 2^{n-1}(x_1 - x_2)(x_2 - x_3) \cdots (x_{2n-1} - x_{2n})(x_{2n} - x_1).$$

We show the above property by induction. It holds for $n = 1$:

$$\mathrm{pf}(A_1) = \mathrm{pf}\left(\begin{bmatrix} 0 & (x_1 - x_2)^2 \\ -(x_1 - x_2)^2 & 0 \end{bmatrix}\right) = (x_1 - x_2)^2 = -(x_1 - x_2)(x_2 - x_1).$$

Given $n > 1$, we assume that the claim holds for any positive integer less than n. By expanding $\mathrm{pf}(A_n)$ along the first row, we have that

$$\mathrm{pf}(A_n) = \sum_{j=2}^{2n} (-1)^j (x_1 - x_j)^2 \, \mathrm{pf}((A_n)_{1,j})$$

where $(A_n)_{1,j}$ is the matrix with both the first and the j-th rows and columns removed. Hence it remains to show that

$$\sum_{j=2}^{2n} (-1)^j (x_1 - x_j)^2 \, \mathrm{pf}((A_n)_{1,j}) = (-1)^n 2^{n-1}(x_1 - x_2)(x_2 - x_3) \cdots (x_{2n-1} - x_{2n})(x_{2n} - x_1) \quad (1)$$

where, by the induction hypothesis,

$$\mathrm{pf}((A_n)_{1,j}) = (-1)^{n-1} 2^{n-2} \cdot \begin{cases} (x_3 - x_4) \cdots (x_{2n-1} - x_{2n})(x_{2n} - x_3) & \text{if } j = 2, \\ (x_2 - x_3) \cdots (x_{j-1} - x_{j+1}) \cdots (x_{2n-1} - x_{2n})(x_{2n} - x_2) & \text{if } 2 < j < 2n, \\ (x_2 - x_3) \cdots (x_{2n-2} - x_{2n-1})(x_{2n-1} - x_2) & \text{if } j = 2n. \end{cases}$$

We may assume that $x_j \neq x_{j+1}$, otherwise both sides of (1) are trivially zero.
It easy to verify that (1) is equivalent to

$$\begin{aligned} \frac{(x_1 - x_2)^2 (x_{2n} - x_3)}{x_2 - x_3} &+ (x_{2n} - x_2) \sum_{j=3}^{2n-1} (-1)^j \frac{(x_1 - x_j)^2 (x_{j-1} - x_{j+1})}{(x_{j-1} - x_j)(x_j - x_{j+1})} \\ &+ \frac{(x_1 - x_{2n})^2 (x_{2n-1} - x_2)}{x_{2n-1} - x_{2n}} = -2(x_1 - x_2)(x_{2n} - x_1). \end{aligned} \quad (2)$$

We have that

$$\begin{aligned}
\sum_{j=3}^{2n-1}(-1)^j\frac{(x_1-x_j)^2(x_{j-1}-x_{j+1})}{(x_{j-1}-x_j)(x_j-x_{j+1})} &= \sum_{j=3}^{2n-1}(-1)^j\frac{(x_1-x_j)^2}{x_{j-1}-x_j}+\sum_{j=3}^{2n-1}(-1)^j\frac{(x_1-x_j)^2}{x_j-x_{j+1}}\\
&= \sum_{j=3}^{2n-1}(-1)^j\frac{(x_1-x_j)^2}{x_{j-1}-x_j}-\sum_{j=4}^{2n}(-1)^j\frac{(x_1-x_{j-1})^2}{x_{j-1}-x_j}\\
&= -\frac{(x_1-x_3)^2}{x_2-x_3}+\sum_{j=4}^{2n-1}(-1)^j(2x_1-x_{j-1}-x_j)-\frac{(x_1-x_{2n-1})^2}{x_{2n-1}-x_{2n}}\\
&= -\frac{(x_1-x_3)^2}{x_2-x_3}+x_{2n-1}-x_3-\frac{(x_1-x_{2n-1})^2}{x_{2n-1}-x_{2n}}.
\end{aligned}$$

Therefore, the left-hand side of (2) is equal to

$$\frac{(x_1-x_2)^2(x_{2n}-x_3)}{x_2-x_3}+(x_{2n}-x_2)\left(-\frac{(x_1-x_3)^2}{x_2-x_3}+x_{2n-1}-x_3-\frac{(x_1-x_{2n-1})^2}{x_{2n-1}-x_{2n}}\right)+\frac{(x_1-x_{2n})^2(x_{2n-1}-x_2)}{x_{2n-1}-x_{2n}}$$

which reduces to $-2(x_1-x_2)(x_{2n}-x_1)$, the right-hand side of (2). □