Problem 12320
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Proposed by E. Treviño (USA).

Consider the grid of n^2 *lattice points* $\{1,\ldots,n\}^2$ *. Let* $S_1(n)$ *be the number of rectangles with corners in the grid that have area equal to a prime integer congruent to* $1 \pmod 4$*. Define* $S_3(n)$ *similarly using primes congruent to* $3 \pmod 4$*. Prove that there is a value* n_0 *such that* $S_1(n) > S_3(n)$ *for* $n \geq n_0$*.*

Solution proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.

Solution. Any rectangle with corners in the grid is of two distinct types: it has horizontal and vertical sides, or its sides are tilted.

i) The area of a rectangle with horizontal side a and vertical side b is equal to $a \cdot b$. Hence the area is a prime number p if and only if $a = p$ and $b = 1$ or $a = 1$ and $b = p$. In the grid $n \times n$, for $p < n$, there are $2(n-1)(n-p)$ copies of such rectangle.

ii) Any rectangle with tilted sides is defined by two orthogonal vectors $s(a,b)$ and $t(-b,a)$ at the bottom vertex, with s,t,a,b positive integers such that $\gcd(a,b) = 1$. Its area is equal to $s \cdot t \cdot (a^2+b^2)$. Note that this kind of rectangle is inscribed in a square with horizontal and vertical sides of length $a+b$. Hence the area is a prime number p if and only if $s = t = 1$ and $a^2 + b^2 = p$. In the grid $n \times n$, for $a + b < n$, there are $(n-a-b)^2$ copies of such rectangle.

Since a prime p can be written as the sums of two squares if and only if $p \equiv 1 \pmod 4$, from the comments above, it follows that

$$S_1(n) = \sum_{\substack{p\equiv 1 \pmod 4 \\ p<n}} 2(n-1)(n-p) + \sum_{\substack{p\equiv 1 \pmod 4 \\ a+b<n, a^2+b^2=p}} (n-a-b)^2,$$

$$S_3(n) = \sum_{\substack{p\equiv 3 \pmod 4 \\ p<n}} 2(n-1)(n-p).$$

By the Prime Number Theorem, for $\alpha \geq 0$,

$$T_\alpha(x) := \sum_{p\leq x} p^\alpha \sim \frac{x^{\alpha+1}}{(\alpha+1)\log(x)},$$

and if the sum is restricted to the primes congruent to 1 modulo 4 or to 3 modulo 4, then, by the Prime Number Theorem for arithmetic progressions, there is an extra factor $1/\varphi(4) = 1/2$ in the asymptotic estimate.
Therefore

$$S_3(n) \sim \sum_{\substack{p\equiv 3 \pmod 4 \\ p<n}} (2n^2 - 2np) \sim n^2 T_0(n) - nT_1(n) = n^2 \frac{n}{\log(n)} - n\frac{n^2}{2\log(n)} = \frac{n^3}{2\log(n)}.$$

As regards $S_1(n)$, since

$$(a+b)^2 \leq (1^2+1^2)(a^2+b^2) = 2p \implies a+b \leq \sqrt{2p},$$

we have that

$$S_1(n) \geq \sum_{\substack{p\equiv 1 \pmod 4 \\ a+b<n, a^2+b^2=p}} (n-a-b)^2 \geq \sum_{\substack{p\equiv 1 \pmod 4 \\ \sqrt{2p}<n}} (n-\sqrt{2p})^2.$$

Moreover

$$\begin{aligned}\sum_{\substack{p\equiv 1 \pmod 4\\ \sqrt{2p}<n}} (n-\sqrt{2p})^2 &= \sum_{\substack{p\equiv 1 \pmod 4\\ p<n^2/2}} (n^2 - 2\sqrt{2}np^{1/2} + 2p)\\ &\sim \frac{n^2}{2}T_0(n^2/2) - \sqrt{2}nT_{1/2}(n^2/2) + T_1(n^2/2)\\ &\sim \frac{n^2}{2}\frac{(n^2/2)}{\log(n^2/2)} - \sqrt{2}n\frac{2(n^2/2)^{3/2}}{3\log(n^2/2)} + \frac{(n^2/2)^2}{2\log(n^2/2)}\\ &\sim \frac{n^4}{\log(n)}\left(\frac{1}{8} - \frac{1}{6} + \frac{1}{16}\right) = \frac{n^4}{48\log(n)}.\end{aligned}$$

Hence, there is some positive constant c such that, for every sufficiently large n,

$$\frac{S_1(n)}{S_3(n)} \geq c\frac{n^4/\log(n)}{n^3/\log(n)} = cn$$

which implies that there is a value n_0 such that $S_1(n) > S_3(n)$ for $n \geq n_0$. □