Problem 12319
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Proposed by M. Bencze (Romania).

Let ABC be a triangle with all angles less than $120°$, and let F be the Fermat point of ABC (the point in the interior that minimizes the sum of the distances to A, B, and C). Prove

$$\frac{FA^4}{AB^2}+\frac{FB^4}{BC^2}+\frac{FC^4}{CA^2}\geq\frac{FA^3+FB^3+FC^3}{FA+FB+FC}.$$

Solution proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.

Solution. Since all angles of ABC are less than $120°$, then F is an interior point of the triangle from which each side subtends an angle of $120°$. Hence, by the law of cosines,

$$AB^2=x^2+xy+y^2,\quad BC^2=y^2+yz+z^2,\quad CA^2=z^2+zx+x^2$$

where $x=FA$, $y=FB$, $z=FC$.
Hence we have to show that

$$\sum_{\text{cyc}}x\cdot\sum_{\text{cyc}}\frac{x^4}{x^2+xy+y^2}\geq\sum_{\text{cyc}}x^3.$$

We have that

$$\begin{aligned}\sum_{\text{cyc}}x\cdot\sum_{\text{cyc}}\frac{x^4}{x^2+xy+y^2}&=\sum_{\text{cyc}}x\cdot\sum_{\text{cyc}}\frac{x(x^3-y^3)+xy^3}{x^2+xy+y^2}\\&=\sum_{\text{cyc}}x\cdot\sum_{\text{cyc}}x(x-y)+\sum_{\text{cyc}}x\cdot\sum_{\text{cyc}}\frac{xy^3}{x^2+xy+y^2}\\&=\sum_{\text{cyc}}x^3-3xyz+\sum_{\text{cyc}}x\cdot\sum_{\text{cyc}}\frac{xy^3}{x^2+xy+y^2}\\&=\sum_{\text{cyc}}x^3+xyz\left(\sum_{\text{cyc}}x\cdot\sum_{\text{cyc}}\frac{y^2}{z(x^2+xy+y^2)}-3\right)\\&\geq\sum_{\text{cyc}}x^3.\end{aligned}$$

The last step is implied by the following fact: by the Cauchy-Schwarz inequality,

$$\sum_{\text{cyc}}z(x^2+xy+y^2)\cdot\sum_{\text{cyc}}\frac{y^2}{z(x^2+xy+y^2)}\geq\Big(\sum_{\text{cyc}}y\Big)^2$$

and, after noting that $\sum_{\text{cyc}}z(x^2+xy+y^2)=\sum_{\text{cyc}}x\cdot\sum_{\text{cyc}}xy$, we find

$$\sum_{\text{cyc}}x\cdot\sum_{\text{cyc}}\frac{y^2}{z(x^2+xy+y^2)}\geq\frac{\left(\sum_{\text{cyc}}y\right)^2}{\sum_{\text{cyc}}xy}=\frac{\sum_{\text{cyc}}y^2}{\sum_{\text{cyc}}xy}+2\geq 1+2=3.$$

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