**Problem 12318**
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Proposed by M. H. Mehrabi (Sweden).

*Let $a$ be a positive real number, and let $S_a$ be the set of functions $f : [-a,a] \to \mathbb{R}$ such that $\int_{-a}^{a}(f(x))^2\,dx = 1$. Let $A(f) = \int_{-a}^{a} f(x)\,dx$, $B(f) = \int_{-a}^{a} xf(x)\,dx$, and $C(f) = \int_{-a}^{a} x^2 f(x)\,dx$.*
*(a) What is $\sup\{A(f)^2 + B(f)^2 : f \in S_a\}$?*
*(b) What is $\sup\{A(f)^2 + B(f)^2 + C(f)^2 : f \in S_a\}$?*

Solution proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.

*Solution.* It is known that the sequence of Legendre polynomials $\{P_n(t)\}_{n\geq 0}$ is an orthogonal system in $L^2(-1,1)$ and

$$\int_{-1}^{1} P_m(t)P_n(t)\,dt = \begin{cases} \frac{2}{2n+1} & \text{if } m = n, \\ 0 & \text{otherwise.} \end{cases}$$

Hence if $g \in L^2(-1,1)$ then, by Parseval's identity,

$$\int_{-1}^{1}(g(t))^2\,dt = \sum_{n=0}^{\infty}\frac{2c_n^2}{2n+1} \quad \text{with} \quad c_n = \frac{2n+1}{2}\int_{-1}^{1} g(t)P_n(t)\,dt.$$

(a) We note that $\sup\{A(f)^2 + B(f)^2 : f \in S_a\}$ is the smallest constant $M$ such that

$$A(f)^2 + B(f)^2 \leq M\|f\|^2 \qquad \forall f \in L^2(-a,a)$$

that is, after letting $x = at$ and $g(t) = f(at)$,

$$a^2\left(\int_{-1}^{1} g(t)\,dt\right)^2 + a^4\left(\int_{-1}^{1} tg(t)\,dt\right)^2 \leq aM\int_{-1}^{1}(g(t))^2\,dt \qquad \forall g \in L^2(-1,1).$$

Since $P_0(t) = 1$ and $P_1(t) = t$, the above inequality is equivalent to

$$a\,(2c_0)^2 + a^3\left(\frac{2c_1}{3}\right)^2 \leq M\sum_{n=0}^{\infty}\frac{2c_n^2}{2n+1}.$$

Hence it suffices to find the smallest constant $M$ such that for all $c_0, c_1 \in \mathbb{R}$,

$$2ac_0^2 + \frac{2a^3c_1^2}{9} \leq M\Big(c_0^2 + \frac{c_1^2}{3}\Big)$$

or

$$c_0^2(M-2a) + \frac{c_1^2}{3}\Big(M - \frac{2a^3}{3}\Big) \geq 0$$

which implies

$$\boxed{M = \max\left\{2a, \frac{2a^3}{3}\right\}}$$

(b) In a similar way as we did before, we have that $\sup\{A(f)^2 + B(f)^2 + C(f)^2 : f \in S_a\}$ is the smallest constant $M$ such that

$$a^2\left(\int_{-1}^{1} g(t)\,dt\right)^2 + a^4\left(\int_{-1}^{1} tg(t)\,dt\right)^2 + a^6\left(\int_{-1}^{1} t^2g(t)\,dt\right)^2 \leq aM\int_{-1}^{1}(g(t))^2\,dt \qquad \forall g \in L^2(-1,1).$$

Since $P_0(t) = 1$, $P_1(t) = t$ and $P_2(t) = (3t^2 - 1)/2$, then $t^2 = (2P_2(t) + P_0(t))/3$ and the above inequality is equivalent to

$$a\,(2c_0)^2 + a^3\left(\frac{2c_1}{3}\right)^2 + a^5\left(\frac{4c_2}{15} + \frac{2c_0}{3}\right)^2 \leq M\sum_{n=0}^{\infty}\frac{2c_n^2}{2n+1}.$$

Therefore it suffices to find the smallest constant $M$ such that for all $c_0, c_1, c_2 \in \mathbb{R}$,

$$2ac_0^2 + \frac{2a^3c_1^2}{9} + \frac{8a^5c_2^2}{225} + \frac{8a^5c_0c_2}{45} + \frac{2a^5c_0^2}{9} \leq M\Big(c_0^2 + \frac{c_1^2}{3} + \frac{c_2^2}{5}\Big)$$

or

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}^t \begin{bmatrix} M - 2a - \frac{2a^5}{9} & 0 & -\frac{4a^5}{45} \\ 0 & \frac{M}{3} - \frac{2a^3}{9} & 0 \\ -\frac{4a^5}{45} & 0 & \frac{M}{5} - \frac{8a^5}{225} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} \geq 0$$

which holds if and only if the eigenvalues $\lambda_0, \lambda_1, \lambda_2$ of the above symmetric matrix are all non-negative. Clearly $\lambda_1 = \frac{1}{3}(M - \frac{2a^3}{3})$, and

$$\begin{cases} \lambda_0 + \lambda_2 = \frac{6}{5}\left(M - \frac{5a}{3} - \frac{29a^5}{135}\right) \\ \lambda_0 \cdot \lambda_2 = \frac{1}{5}\left(M^2 - 2(a + \frac{a^5}{5})M + \frac{16a^6}{45}\right) = \frac{1}{5}(M - m_-)(M - m_+) \end{cases}$$

where $m_\pm = a + \frac{a^5}{5} \pm a\sqrt{1 + \frac{2a^4}{45} + \frac{a^8}{25}}$. It follows that the smallest constant $M$ such that all the eigenvalues $\lambda_0, \lambda_1, \lambda_2$ are non-negative is given by

$$M = \max\left\{\frac{2a^3}{3}, \frac{5a}{3} + \frac{29a^5}{135}, m_+\right\}$$

that is

$$\boxed{M = m_+ = a + \frac{a^5}{5} + a\sqrt{1 + \frac{2a^4}{45} + \frac{a^8}{25}}}$$

because it can be easily verified that for $a > 0$,

$$\frac{2a^3}{3} < \frac{5a}{3} + \frac{29a^5}{135} < m_+.$$

$\square$