Problem 12317
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Proposed by S. Stewart (Saudi Arabia).

Prove

$$\int_0^{\pi/2} \frac{\sin(4x)}{\log(\tan(x))}\,dx = -\frac{14\zeta(3)}{\pi^2}.$$

Solution proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.

Solution. We have that

$$\begin{aligned}
\int_0^{\pi/2} \frac{\sin(4x)}{\log(\tan(x))}\,dx &= 2\int_0^{\pi/2} \frac{\sin(2x)\cos(2x)}{\log(\tan(x))}\,dx \\
&= 2\int_0^{+\infty} \frac{\frac{2t}{1+t^2}\frac{1-t^2}{1+t^2}}{\log(t)}\,\frac{dt}{1+t^2} \qquad [t=\tan(x)] \\
&= 4\int_0^{+\infty} \frac{t(1-t^2)}{(1+t^2)^3\log(t)}\,dt \\
&= -4\int_0^{+\infty} \frac{s-1}{(1+s)^3\log(s)}\,ds \qquad [s=t^2] \\
&= -4I(1)
\end{aligned}$$

where

$$I(u) = \int_0^{+\infty} \frac{s^u-1}{(1+s)^3\log(s)}\,ds.$$

By the Euler's reflection formula, for $u\in(0,1)$,

$$\begin{aligned}
I'(u) &= \int_0^{+\infty} \frac{s^u}{(1+s)^3}\,ds \overset{w=\frac{1}{1+s}}{=} \int_0^1 w^{1-u}(1-w)^u\,dw = B(2-u,u+1) \\
&= \frac{\Gamma(2-u)\Gamma(u+1)}{\Gamma(3)} = \frac{u(1-u)}{2}\,\Gamma(u)\Gamma(1-u) = \frac{\pi u(1-u)}{2\sin(\pi u)}.
\end{aligned}$$

Therefore

$$\begin{aligned}
I(1) &= \int_0^1 I'(u)\,du = \frac{\pi}{2}\int_0^1 \frac{u(1-u)}{\sin(\pi u)}\,du = \pi i\int_0^1 \frac{u(1-u)}{e^{i\pi u}-e^{-i\pi u}}\,du \\
&= \pi i\sum_{k=0}^{\infty}\int_0^1 u(1-u)e^{-i(2k+1)\pi u}\,du = \frac{4}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3} = \frac{4}{\pi^2}\zeta(3)\left(1-\frac{1}{8}\right) = \frac{7\zeta(3)}{2\pi^2}.
\end{aligned}$$

Finally,

$$\int_0^{\pi/2} \frac{\sin(4x)}{\log(\tan(x))}\,dx = -4I(1) = -4\frac{7\zeta(3)}{2\pi^2} = -14\frac{\zeta(3)}{\pi^2}.$$

□