Problem 12316
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For each i in $\{1, 2, \ldots, C\}$, we have $2i$ coins with color i. Place these $C(C+1)$ coins in a line. A move consists of the transposition of two adjacent coins. Let m be the minimum number of moves required to reach a configuration where all coins of the same color are together in a run of consecutive coins. Show that

$$\frac{(C-1)C(C+1)(3C+2)}{12}.$$

is the maximum value of m over all initial configurations.

Solution proposed by Luciano Gualà (University of Rome Tor Vergata, Rome, Italy), Stefano Leucci (University of L'Aquila, L'Aquila, Italy), and Roberto Tauraso (University of Rome Tor Vergata, Rome, Italy).

Solution. Let $f(C)$ be the minimum number of moves needed to reach the required configuration. If $C > 1$ and the $2C$ coins $\bullet$ of color C are placed along the line as shown below,

$$\underbrace{\circ\ldots\circ}_{n_0}\bullet\underbrace{\circ\ldots\circ}_{n_1}\bullet\underbrace{\circ\ldots\circ}_{n_2}\bullet\circ\ldots\circ\bullet\underbrace{\circ\ldots\circ}_{n_{2C-1}}\bullet\underbrace{\circ\ldots\circ}_{n_{2C}}$$

then we need

$$n_0 + (n_0 + n_1) + (n_0 + n_1 + n_2) + \cdots + (n_0 + n_1 + n_2 + \cdots + n_{2C-1}) = \sum_{k=0}^{2C-1}(2C-k)n_k$$

adjacent transpositions in order to move the $2C$ coins to the far left side.
Similarly, in order to move the $2C$ coins to the far right side, we need

$$n_{2C} + (n_{2C} + n_{2C-1}) + (n_{2C} + n_{2C-1} + n_{2C-2}) + \cdots + (n_{2C} + n_{2C-1} + n_{2C-2} + \cdots + n_1) = \sum_{k=1}^{2C} k n_k$$

adjacent transpositions.
Since the smallest of those two numbers is less or equal to their arithmetic mean, which is

$$\frac{1}{2}\left(\sum_{k=0}^{2C-1}(2C-k)n_k + \sum_{k=1}^{2C} k n_k\right) = \frac{2C}{2}\sum_{k=0}^{2C} n_k = C\cdot(C(C+1)-2C) = C^2(C-1),$$

it follows that

$$f(C) \leq f(C-1) + C^2(C-1).$$

Noting that $f(1)$ is trivially zero, by induction we find

$$f(C) = \sum_{i=2}^{C}(f(i) - f(i-1)) \leq \sum_{i=2}^{C} i^2(i-1) = \frac{(C-1)C(C+1)(3C+2)}{12}.$$

It remains to show that there is an arrangement of coins where the number of adjacent transpositions needed to group all the coins of the same color together is at least $(C-1)C(C+1)(3C+2)/12$. We claim that such *worst case* is given by placing along the line, from left to right, 1 coin of color 1, 2 coins of color 2, ..., C coins of color C, and then mirroring the arrangement so far obtained. For example, for $C = 4$, we have

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where this time an integer i represents a coin of color i.
Fix any permutation P of the colors $1, \ldots, C$. Given any arrangement, the *ordered* pair of positions $\langle i, j\rangle$ is an *inversion* if $i < j$, the color c_i of the coin at position i differs from the color c_j of the coin

at position j, and c_i appears after c_j in P. Since one adjacent transposition reduces the number of inversions by at most one, the number of inversions is a lower bound for the number of adjacent transpositions needed to arrange the coins into groups of colors in the order given by P.
Going back to the claimed worst case, we note that regardless of the choice of P, the number of inversions of our worst-case arrangement is always the same. Indeed, for any pair of coins of different colors at positions i, j with $i < j$, we have that $\langle i, j \rangle$ is an inversion if and only if the symmetric pair $\langle C(C+1) - j, C(C+1) - i \rangle$ is not an inversion. Hence the number of inversions for the worst case is half of the number of *unordered* pairs of coins of different colors, i.e.

$$\frac{1}{2} \sum_{1 \leq i < j \leq C} (2i)(2j) = 2 \sum_{1 \leq i < j \leq C} ij = \left(\sum_{i=1}^{C} i \right)^2 - \sum_{i=1}^{C} i^2 = \frac{(C-1)C(C+1)(3C+2)}{12}$$

and we are done. □