Problem 12315
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Suppose $a_{i,j} \in [0,1]$ *for* $1 \le i \le n$ *and* $1 \le j \le n$. *Let* B *be the* $n \times n$ *matrix with* (i,j)*-entry* $b_{i,j}$ *defined by* $b_{i,j} = a_{i,j}$ *when* $j \ne i-1$ *and* $b_{i,j} = -\sum_{k=1}^{n} a_{i,k}$ *when* $j = i-1$.
(a) Evaluate $\det(B)$ *in the case where* $a_{i,j} = 1$ *for all* i *and* j.
(b) Show that the value in part (a) is the maximum possible value of $\det(B)$.
(c) Show that $\det(B) \ge 0$ *in all cases.*

Solution proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.

Solution. Let us consider the $(n+1)\times(n+1)$ matrix

$$T_{n+1} = \begin{bmatrix} \sum_{k\neq 1} x_{1,k} & -x_{1,2} & -x_{1,3} & \dots & -x_{1,n} & -x_{1,n+1} \\ -x_{2,1} & \sum_{k\neq 2} x_{2,k} & -x_{2,3} & \dots & -x_{2,n} & -x_{2,n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -x_{n,1} & -x_{n,2} & -x_{n,3} & \dots & \sum_{k\neq n} x_{n,k} & -x_{n,n+1} \\ -x_{n+1,1} & -x_{n+1,2} & -x_{n+1,3} & \dots & -x_{n+1,n} & \sum_{k\neq n+1} x_{n+1,k} \end{bmatrix}$$

then, by the Matrix Tree Theorem, the determinant of the submatrix of T_{n+1} obtained by deleting the $(n+1)$-th row and the $(n+1)$-th column is the generating function f_{n+1} of all the trees on $n+1$ vertices labeled $1, 2, \dots, n, n+1$ with edges directed towards the root $n+1$. By Cayley's formula, f_{n+1} is a sum of $(n+1)^{n-1}$ monomials. Each monomial is associated to a tree and it is the product of n variables $x_{i,j}$ for all its directed edges from vertex i to vertex j. For instance, for $n = 2$,

$$f_3 = \det \begin{bmatrix} x_{1,2} + x_{1,3} & -x_{1,2} \\ -x_{2,1} & x_{2,1} + x_{2,3} \end{bmatrix} = x_{1,2}x_{2,3} + x_{1,3}x_{2,1} + x_{1,3}x_{2,3}.$$

Let P_n be the $n \times n$ matrix given by

$$\begin{bmatrix} 1 & -1 & 0 & \dots & 0 \\ 1 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & -1 \\ 1 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

It easy to verify that $\det(P_n) = 1$. Then the matrix product BP_n is equal to

$$\begin{bmatrix} \sum_k a_{1,k} & -a_{1,1} & -a_{1,2} & \dots & -a_{1,n} \\ -a_{2,1} & \sum_k a_{2,k} & -a_{2,2} & \dots & -a_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n,n-1} & -a_{n,1} & -a_{n,2} & \dots & \sum_k a_{n,k} \end{bmatrix} = \begin{bmatrix} \sum_{k\neq 1} x_{1,k} & -x_{1,2} & -x_{1,3} & \dots & -x_{1,n} \\ -x_{2,1} & \sum_{k\neq 2} x_{2,k} & -x_{2,3} & \dots & -x_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -x_{n,1} & -x_{n,2} & -x_{n,3} & \dots & \sum_{k\neq n} x_{n,k} \end{bmatrix}$$

where the matrix on the right is T_{n+1} without the $(n+1)$-th row and the $(n+1)$-th column. Such equality gives a bijection between the variables $x_{i,j}$ and the variables $a_{k,l}$. Hence

$$\det(B) = \det(BP_n) = f_{n+1}.$$

Finally, since each monomial of f_{n+1} is a product of variables $x_{i,j} = a_{k,l} \in [0,1]$, it follows that also its value belongs to $[0,1]$, and therefore

$$0 \le \det(B) \le (n+1)^{n-1}$$

where the maximum value $(n+1)^{n-1}$ is attained by setting all variables to 1. □