Problem 12314
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Proposed by G. Galperin and Y. J. Ionin (USA).

Let n, m, and k be positive integers with $k \leq n-1$. Consider n devices each of which can be in any of m states denoted $0, 1, \dots, m-1$. A move consists of selecting a set of k devices and adding $1 \pmod m$ to each of their states. Prove that for any n, m, k as specified and any initial states of the n devices, there exists a sequence of moves that leaves each device in the state 0 or 1.

Solution proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.

Solution. We first show that it suffices to solve the case when $n = k+1$. In fact, if $n > k+1$ then we make the following moves. We add 1 to the state of the $(k+2)$-th device and also to each state of the first $k-1$ devices. We carry on in this way until the state of the $(k+2)$-th device becomes 0 modulo m. After repeating this procedure also for the devices from $k+3$ to n, we find that their states are all reduced to 0 and it remains to solve the problem for the first $k+1$ devices.

Let us consider the case $n = k+1$. Let $\mathbf{w} = (w_1, w_2, \dots, w_{k+1})$ be the vector where each w_j is the initial state of j-th device. For $j = 1, \dots, k+1$, let $\mathbf{v}_j$ be the $(k+1)$-dimensional vector which has 0 as j-th entry and 1s elsewhere. We apply, in order, three groups of moves:

i) Add $\left(m - \lfloor \frac{t}{k} \rfloor\right)$ times $\mathbf{v}_1$ to $\mathbf{w}$, where $t = \sum_{j=1}^{k+1} w_j \pmod m$ is the initial state-sum. Then the new state-sum is

$$\sum_{j=1}^{k+1} w_j = t + \left(m - \left\lfloor \frac{t}{k} \right\rfloor\right) k = r \pmod m$$

where r is the remainder of the division of t by k. Note that $0 \leq r < k$.

ii) For j from 2 to $k+1$, add w_j times $\mathbf{v}_j + (m-1)\mathbf{v}_1$ to $\mathbf{w}$ modulo m. After the j-th step, w_j becomes 0, w_1 increases by w_j, whereas the state-sum remains the same. At the end of this second stage, we have that $\mathbf{w} = (r, 0, \dots, 0)$.

iii) For j from 2 to r, add $\mathbf{v}_1 + (m-1)\mathbf{v}_j$ to $\mathbf{w}$ modulo m. After the j-th step, w_j becomes 1, whereas w_1 decreases by 1. At the end of this third stage,

$$\mathbf{w} = (\underbrace{1, \dots, 1}_{r}, 0, \dots, 0)$$

and we are done. $\square$