Problem 12313
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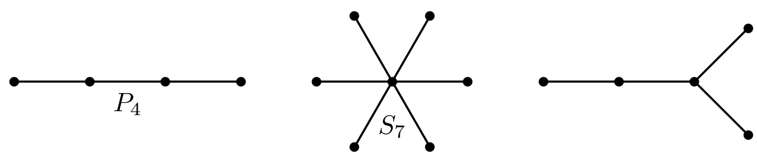
Proposed by D. B. West (USA).

For all $n \in \mathbb{N}$, determine all n-vertex trees having the property that the connected $(n-2)$-vertex subgraphs that can be obtained by deleting two vertices are pairwise isomorphic.

Solution proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.

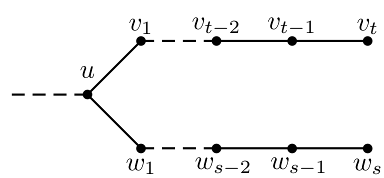
Solution. We will prove that the required property holds if and only if the tree is a path graph P_n, a star graph S_n, or the 5-vertex tree pictured below on the right:



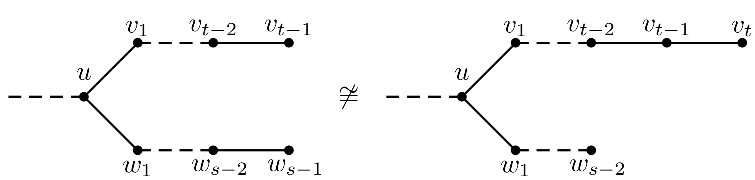
It is easy to verify that such trees satisfy the property, so we have to show that for any other tree T we can find two pairs of vertices $\{v_1, v_2\}$ and $\{v_3, v_4\}$ such that the subtrees $T - \{v_1, v_2\}$ and $T - \{v_3, v_4\}$ are not isomorphic.
We may assume that T has $n \geq 6$ vertices with at least a vertex of degree ≥ 3 (T is not a path). By the handshaking lemma, the number of leaves n_1 exceeds $n_{\geq 3}$, the number of vertices of degree 3 or more,

$$2(n_1 + n_2 + n_{\geq 3} - 1) = 2(n-1) = \sum_{v \in T} \deg(v) \geq n_1 + 2n_2 + 3n_{\geq 3} \implies n_1 \geq 2 + n_{\geq 3}.$$

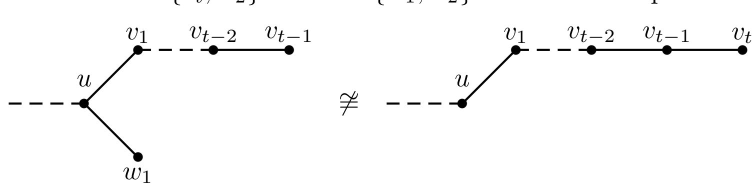
Then, since any leaf has a unique closest vertex of degree ≥ 3, there is a vertex u of degree ≥ 3 which is incident to at least two paths of length $t \geq s \geq 1$:



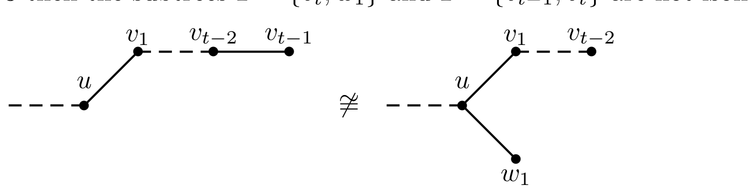
If $s \geq 3$ then the subtrees $T - \{v_t, w_s\}$ and $T - \{w_{s-1}, w_s\}$ are not isomorphic:



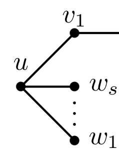
If $s = 2$ then the subtrees $T - \{v_t, w_2\}$ and $T - \{w_1, w_2\}$ are not isomorphic:



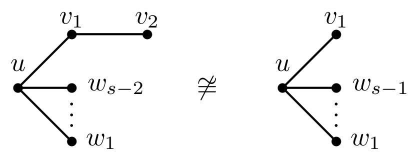
If $s = 1$ and $t \geq 3$ then the subtrees $T - \{v_t, w_1\}$ and $T - \{v_{t-1}, v_t\}$ are not isomorphic:



It remains to consider the cases when all vertices of degree ≥ 3 which are incident to at least one path, have all those paths of length 1 with at most one path of length 2.
If there is a unique vertex u of degree ≥ 3 then u is incident to one path of length 2 and $s \geq 3$ paths of length 1 (recall that $n \geq 6$ and T is not a star graph):

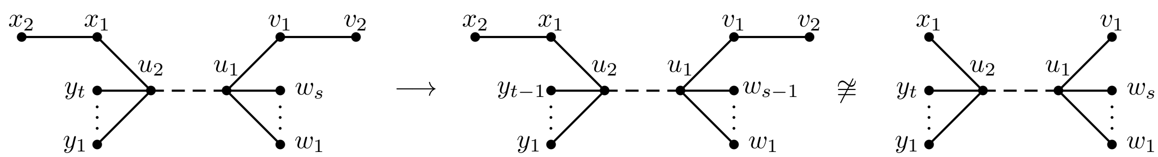


Then the subtrees $T - \{w_{s-1}, w_s\}$ and $T - \{w_s, v_2\}$ are not isomorphic:

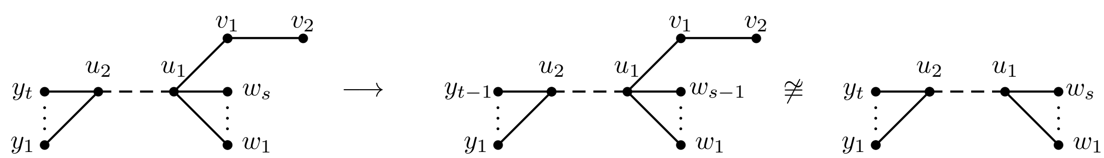


Finally, if $n_{\geq 3} \geq 2$ then there are two vertices u_1 and u_2 of degree ≥ 3 which are incident to at least two paths each. We have three distinct cases.

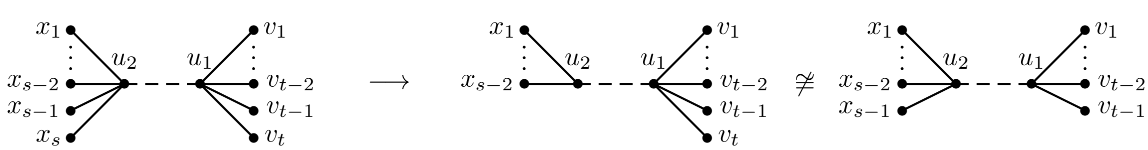
i) If u_1 and u_2 are both incident to a path of length 2 and to $s, t \geq 1$ paths of length 1 respectively then the subtrees $T - \{y_t, w_s\}$ and $T - \{x_2, v_2\}$ are not isomorphic:



ii) If u_1 and u_2 are incident to $s \geq 1$ and $t \geq 2$ paths of length 1 respectively, but just u_1 is incident to a path of length 2 then the subtrees $T - \{y_t, w_s\}$ and $T - \{v_1, v_2\}$ are not isomorphic:



iii) If both u_1 and u_2 are not incident to a path of length 2, u_1 is incident to t leaves and u_2 is incident to s leaves with $2 \leq s \leq t$ then $T - \{x_{s-1}, x_s\}$ and $T - \{x_s, v_t\}$ are not isomorphic:



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