Problem 12312
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Find all continuous functions $f : [0, \infty) \to \mathbb{R}$ *such that, for all positive* x,

$$f(x)\left(f(x) - \frac{1}{x}\int_0^x f(t)\,dt\right) \geq (f(x) - 1)^2.$$

Solution proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.

Solution. We will prove that there is only one continuous function such that the inequality holds: the constant function 1.

We first note that by L'Hospital's rule

$$\lim_{x\to 0^+} \frac{1}{x}\int_0^x f(t)\,dt = \lim_{x\to 0^+} f(x) = f(0)$$

and therefore by taking the limit as $x \to 0^+$ of both sides of the inequality we find $0 \geq (f(0) - 1)^2$, which implies $f(0) = 1$. Furthermore the inequality can be written as

$$(1 + F(x) + xF'(x))\,(1 + F(x) + xF'(x) - (1 + F(x)) \geq (F(x) + xF'(x))^2$$

which simplifies to

$$xF'(x)(1 - F(x)) \geq F(x)^2 \tag{1}$$

where

$$F(x) := \frac{1}{x}\int_0^x f(t)\,dt - 1$$

is a C^1 function in $(0, +\infty)$ with $D(xF(x)) = F(x) + xF'(x) = f(x) - 1$.
We claim that F' is identically 0 in $(0, +\infty)$ and therefore, due to

$$\lim_{x\to 0^+} F(x) = \lim_{x\to 0^+} f(x) - 1 = f(0) - 1 = 0,$$

it follows that F is identically 0 and finally we may conclude that f is identically 1.

Assume by contradiction that F is not identically 0 then, by continuity, there exist $x_0 \geq 0$ and $r > 0$ such that $F(x) \neq 0$ and $(1 - F(x)) > 0$ for all $x \in (x_0, x_0 + r)$. By (1), we find that $F'(x) > 0$ for all $x \in (x_0, x_0 + r)$. Actually $F'(x) > 0$ for all $x \in (x_0, +\infty)$, otherwise there would be a point $x_1 > x_0$, such that $F'(x_1) = 0$ and F is positive and strictly increasing in $(x_0, x_1]$, whereas by (1) we obtain $F(x_1) = 0$. Moreover there is no $x > 0$ such that $F(x) = 1$ otherwise, again by (1), $F(x) = 0$ getting a contradiction. Therefore F has a finite limit $L \in (0, 1]$ as $x \to +\infty$. Then

$$\lim_{x\to+\infty} \frac{F(x)^2}{1 - F(x)} = \begin{cases} \frac{L^2}{1-L} & \text{if } 0 < L < 1, \\ +\infty & \text{if } L = 1, \end{cases}$$

which implies that there are $x_2 > x_0$ and $M > 0$ such that $\frac{F(x)^2}{1-F(x)} \geq M$ for all $x > x_2$.
Thus, by (1), we have that $F'(x) \geq \frac{M}{x}$ for all $x > x_2$, and as $x \to +\infty$, we find

$$F(x) = F(x_2) + \int_{x_2}^x F'(t)\,dt \geq F(x_2) + M\int_{x_2}^x \frac{1}{t}\,dt = F(x_2) + M\ln(x/x_2) \to +\infty$$

against the fact that $F(x) \to L \leq 1$. $\square$