**Problem 12311**
(American Mathematical Monthly, Vol.129, March 2022)

Proposed by H. Ohtsuka (Japan).

*Let $m$ and $n$ be positive integers, and let $r, x_1, x_2, \ldots, x_n$ be positive real numbers.*
*(a) Prove* $\prod_{j=0}^{m}\left(\frac{1}{n}\sum_{k=1}^{n} x_k^j\right)^r \geq \left(\frac{1}{n}\sum_{k=1}^{n} x_k^r\right)^{\binom{m+1}{2}}$ *when* $r \leq m/2$.
*(b) Prove* $\prod_{j=0}^{m}\left(\frac{1}{n}\sum_{k=1}^{n} x_k^j\right)^r \leq \left(\frac{1}{n}\sum_{k=1}^{n} x_k^r\right)^{\binom{m+1}{2}}$ *when* $r \geq m$.

Solution proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.

*Solution.* Let $M_t(\mathbf{x}) = \left(\frac{1}{n}\sum_{k=1}^{n} x_k^t\right)^{1/t}$ be the *power mean* with exponent $t$, then the so-called *Power Mean Inequality* holds

$$M_t(\mathbf{x}) \leq M_s(\mathbf{x}) \qquad \text{for } t \leq s.$$

(a) By Cauchy–Schwarz inequality,

$$\sum_{k=1}^{n} x_k^j \cdot \sum_{k=1}^{n} x_k^{m-j} = \sum_{k=1}^{n}(x_k^{\frac{j}{2}})^2 \cdot \sum_{k=1}^{n}(x_k^{\frac{m-j}{2}})^2 \geq \left(\sum_{k=1}^{n} x_k^{\frac{j}{2}+\frac{m-j}{2}}\right)^2 = \left(\sum_{k=1}^{n} x_k^{\frac{m}{2}}\right)^2$$

and therefore

$$\left(\prod_{j=0}^{m}\sum_{k=1}^{n} x_k^j\right)^2 = \prod_{j=0}^{m}\left(\sum_{k=1}^{n} x_k^j \cdot \sum_{k=1}^{n} x_k^{m-j}\right) \geq \left(\sum_{k=1}^{n} x_k^{\frac{m}{2}}\right)^{2(m+1)}$$

which implies

$$\prod_{j=0}^{m}\left(\frac{1}{n}\sum_{k=1}^{n} x_k^j\right) \geq \left(\frac{1}{n}\sum_{k=1}^{n} x_k^{\frac{m}{2}}\right)^{m+1} = \left(M_{\frac{m}{2}}(\mathbf{x})\right)^{\binom{m+1}{2}}.$$

Finally, since $r \leq m/2$, by the Power Mean Inequality,

$$\prod_{j=0}^{m}\left(\frac{1}{n}\sum_{k=1}^{n} x_k^j\right)^r \geq \left(M_{\frac{m}{2}}(\mathbf{x})\right)^{r\binom{m+1}{2}} \geq \left(M_r(\mathbf{x})\right)^{r\binom{m+1}{2}} \geq \left(\frac{1}{n}\sum_{k=1}^{n} x_k^r\right)^{\binom{m+1}{2}}.$$

(b) We have that $r \geq m \geq j \geq 0$ and by the Power Mean Inequality,

$$\prod_{j=0}^{m}\left(\frac{1}{n}\sum_{k=1}^{n} x_k^j\right)^r = \prod_{j=0}^{m}\left(M_j(\mathbf{x})\right)^{rj} \leq \prod_{j=0}^{m}\left(M_r(\mathbf{x})\right)^{rj} = \left(M_r(\mathbf{x})\right)^{r\sum_{j=0}^{m} j} = \left(\frac{1}{n}\sum_{k=1}^{n} x_k^r\right)^{\binom{m+1}{2}}.$$

□