**Problem 12309**
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Proposed by J. DeVincentis, T. C. Occhipinti, and D. J. Velleman (USA).

*Consider a square grid that is infinite in all directions, with tiles placed on finitely many squares of the grid. Two grid squares are called adjacent if they share an edge. There are two types of legal moves:*

*(a) If two tiles are on adjacent squares, then they can both be removed.*

*(b) If a tile is on a square and all adjacent squares are unoccupied, then the tile can be removed with four new tiles then placed on the four adjacent squares.*

*For which initial configurations is it possible to eliminate all tiles from the grid?*

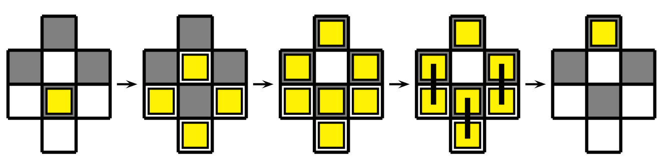
Solution proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.

*Solution.* We consider the grid as an infinite checkerboard with alternating black and white squares. Our claim is as follows:

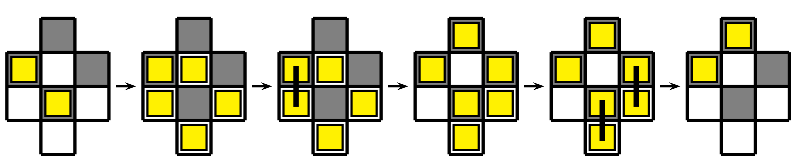
*For a given initial configuration it is possible to eliminate all the tiles if and only if the number of tiles on black squares is congruent modulo* 5 *to the number of tiles on white squares.*

The condition is necessary because by applying the legal moves the difference between the number of tiles on black squares and the number of tiles on white squares modulo 5 is invariant: after move (a) the two numbers decrease both by 1, and after move (b) the difference increases or decreases by $4-(-1)=5$. Moreover, when all the tiles are removed such difference is zero.

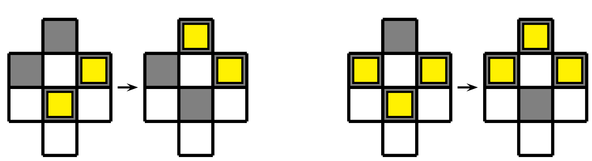
Now we assume that initial configuration satisfies the above condition and we prove that all the tiles can be eliminated. We first remove all the tiles on white squares: if a tile on a white square is adjacent to another tile then it can be eliminated by making move (a), otherwise we remove it by applying move (b). After this first step, the tiles are all on black squares, and, due to the condition, the total number of tiles is a multiple of 5.
We note that any tile on a black square can be moved by two squares horizontally or vertically to an unoccupied black square in any direction. We denote this composite move by (c). For instance, in order to move a tile upward we may apply the following moves:



It is easy to verify that move (c) is not affected by the presence of other tiles on black squares:

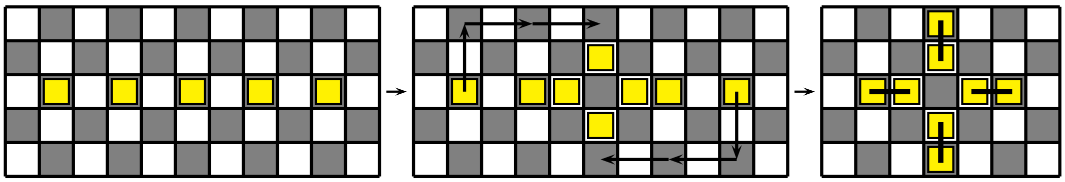


and, in a similar way, by a proper sequence of moves (a) and (b), we obtain

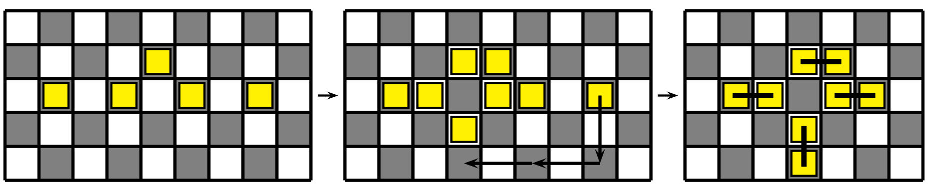


As a second step, by applying move (c) repeatedly, we are able to arrange the tiles, all on black squares, in groups of 5, all along two adjacent horizontal lines (moving the tiles by a 2-step they can't change their parity). Furthermore, each group is exactly one of those three types:

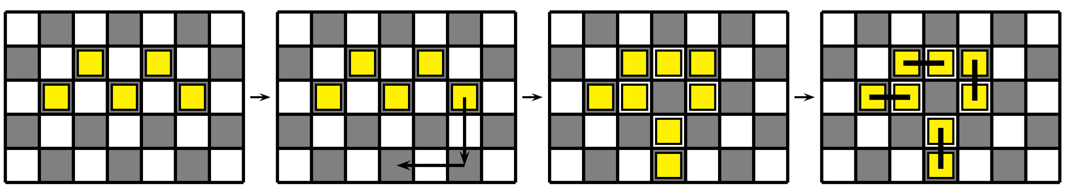
i) 5 tiles in a single row,



ii) 4 tiles in a row and 1 in the another,



iii) 3 tiles in a row and 2 in the another,



Since we are able to eliminate all the tiles in each group, we are done. □