Problem 12308
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Proposed by C. Lupu (China).

What is the minimum value of $\int_0^1 (f'(x))^2\,dx$ *over all continuously differentiable functions* $f:[0,1]\to\mathbb{R}$ *such that* $\int_0^1 f(x)\,dx = \int_0^1 x^2 f(x)\,dx = 1$*?*

Solution proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.

Solution. We show that the minimum value is $\dfrac{105}{2}$.

Let $f \in C^1([0,1])$ such that $\int_0^1 f(x)\,dx = \int_0^1 x^2 f(x)\,dx = 1$. Then

$$\begin{aligned}\int_0^1 x(1-x^2)f'(x)\,dx &= \left[x(1-x^2)f(x)\right]_0^1 - \int_0^1 (1-3x^2)f(x)\,dx\\ &= 0 - \int_0^1 f(x)dx + 3\int_0^1 x^2 f(x)\,dx = -1+3 = 2.\end{aligned}$$

Moreover, by Cauchy–Schwarz inequality,

$$\left(\int_0^1 x(1-x^2)f'(x)\,dx\right)^2 \leq \left(\int_0^1 (x(1-x^2))^2\,dx\right)\left(\int_0^1 (f'(x))^2\,dx\right)$$

which implies

$$\int_0^1 (f'(x))^2\,dx \geq \frac{\left(\int_0^1 x(1-x^2)f'(x)\,dx\right)^2}{\int_0^1 (x(1-x^2))^2\,dx} = \frac{2^2}{8/105} = \frac{105}{2}.$$

The value $\frac{105}{2}$ is attained for the polynomial:

$$f_{\min}(x) = -\frac{105x^4 - 210x^2 + 33}{16}$$

where $\int_0^1 f_{\min}(x)\,dx = \int_0^1 x^2 f_{\min}(x)\,dx = 1$. □