

Problem 12306

(American Mathematical Monthly, Vol.129, February 2022)

Proposed by A. Awasthi (India).

For a positive integer n , evaluate

$$\sum_{a|n} \phi(a) \ln(a) + \sum_{a|n} \sum_{b|(n/a)} \phi(a) \Lambda(b)$$

where ϕ is the Euler phi function and Λ is the von Mangoldt function.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We have that

$$\begin{aligned} \sum_{a|n} \phi(a) \ln(a) + \sum_{a|n} \sum_{b|(n/a)} \phi(a) \Lambda(b) &= \sum_{a|n} \phi(a) \ln(a) + \sum_{a|n} \phi(a) \sum_{b|(n/a)} \Lambda(b) \\ &= \sum_{a|n} \phi(a) \ln(a) + \sum_{a|n} \phi(a) \ln(n/a) \\ &= \sum_{a|n} \phi(a) (\ln(a) + \ln(n/a)) \\ &= \ln(n) \sum_{a|n} \phi(a) = n \ln(n) \end{aligned}$$

where we applied the following two well-known identities

$$\sum_{d|n} \phi(d) = n \quad \text{and} \quad \sum_{d|n} \Lambda(d) = \ln(n).$$

□