Problem 12305
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Proposed by S. Sharma (India).

Prove

$$\int_0^1 \frac{x-1-x\ln(x)}{x\ln(x)-x\ln^2(x)}\,dx = \gamma$$

where γ is the Euler-Mascheroni constant.

Solution proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.

Solution. For $x \in (0,1)$, we have that

$$\frac{x-1-x\ln(x)}{x\ln(x)-x\ln^2(x)} - \left(\frac{1}{\ln(x)}+\frac{1}{1-x}\right) = -\frac{1}{1-x} - \frac{1}{x\ln(x)} - \frac{1}{x(1-\ln(x))} = f'(x)$$

with $f(x) = \ln\left(\frac{(1-x)(1-\ln(x))}{-\ln(x)}\right)$. Therefore, since

$$\lim_{x\to 0^+} f(x) = 0 \quad \text{and} \quad \lim_{x\to 1^-} f(x) = \lim_{t\to 0^+} \ln\left(\frac{t(1-\ln(1-t))}{-\ln(1-t)}\right) = 0,$$

it follows that

$$\begin{aligned}
\int_0^1 \frac{x-1-x\ln(x)}{x\ln(x)-x\ln^2(x)}\,dx &= \int_0^1 \left(\frac{1}{\ln(x)}+\frac{1}{1-x}\right)dx \\
&= \int_0^{+\infty} \left(\frac{e^{-t}}{1-e^{-t}} - \frac{e^{-t}}{t}\right)dt \qquad (x=e^{-t}) \\
&= \int_0^{+\infty} \sum_{n=1}^{\infty} \left(e^{-nt} - \frac{e^{-nt}-e^{-(n+1)t}}{t}\right)dt \\
&= \sum_{n=1}^{\infty} \int_0^{+\infty} \left(e^{-nt} - \frac{e^{-nt}-e^{-(n+1)t}}{t}\right)dt \qquad \left(0 < \frac{1-e^{-t}}{t} < 1 \text{ for } t>0\right) \\
&= \sum_{n=1}^{\infty} \left(\int_0^{+\infty} e^{-nt}\,dt - \int_0^{+\infty} \frac{e^{-nt}-e^{-(n+1)t}}{t}\,dt\right) \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{n} - \ln\left(\frac{n+1}{n}\right)\right) \qquad \text{(Frullani's Theorem)} \\
&= \lim_{N\to\infty} \left(\sum_{n=1}^{N} \frac{1}{n} - \ln(N+1)\right) = \gamma.
\end{aligned}$$

□