Problem 12304
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Proposed by M. Bataille (France).

Let m and n be positive integers with $m < n$. Prove

$$\left(\sum_{k=0}^{m}\binom{m}{k}\frac{(-1)^k}{n-k}\right)\left(\sum_{k=0}^{m}\binom{n}{k}\frac{(-1)^k}{k+1}\right)=\sum_{k=0}^{m}\binom{m}{k}\frac{(-1)^k}{(n-k)(k+1)}.$$

Solution proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.

Solution. Since $\frac{1}{(n-k)(k+1)}=\frac{1}{n+1}\left(\frac{1}{n-k}+\frac{1}{k+1}\right)$, the identity is equivalent to

$$\left(\sum_{k=0}^{m}\binom{m}{k}\frac{(-1)^k}{n-k}\right)\left(\sum_{k=0}^{m}\binom{n}{k}\frac{(-1)^k}{k+1}-\frac{1}{n+1}\right)=\frac{1}{n+1}\sum_{k=0}^{m}\binom{m}{k}\frac{(-1)^k}{k+1}.$$

By using the the known identities

$$\sum_{k=0}^{m}\binom{m}{k}\frac{(-1)^k}{n-k}=\frac{(-1)^m}{(n-m)\binom{n}{m}}\quad\text{and}\quad\sum_{k=0}^{m}\binom{n}{k}\frac{(-1)^k}{k+1}=\frac{(-1)^m(n-m)}{(n+1)(m+1)}\binom{n}{m}+\frac{1}{n+1},\tag{1}$$

it becomes

$$\frac{(-1)^m}{(n-m)\binom{n}{m}}\cdot\frac{(-1)^m(n-m)}{(n+1)(m+1)}\binom{n}{m}=\frac{1}{n+1}\cdot\frac{1}{m+1}$$

which trivially holds.

For the sake of completeness, we include the proofs of the two identities given in (1).
For the first one, we apply partial fraction decomposition and then we let $x=n$,

$$\frac{m!(-1)^m}{x(x-1)\cdots(x-m)}=\sum_{k=0}^{m}\frac{1}{x-k}\cdot\frac{m!(-1)^k}{k(k-1)\cdots1\cdot1\cdots(m-k)}=\sum_{k=0}^{m}\binom{m}{k}\frac{(-1)^k}{x-k}.$$

As regards the second identity, for $0\le m\le n$,

$$\begin{aligned}\sum_{k=0}^{m}\binom{n}{k}\frac{(-1)^k}{k+1}&=\frac{1}{n+1}\sum_{k=0}^{m}(-1)^k\binom{n+1}{k+1}=\frac{1}{n+1}\sum_{k=0}^{m}\left((-1)^k\binom{n}{k+1}-(-1)^{k-1}\binom{n}{k}\right)\\&=\frac{1}{n+1}\left((-1)^m\binom{n}{m+1}+1\right)=\frac{(-1)^m(n-m)}{(n+1)(m+1)}\binom{n}{m}+\frac{1}{n+1}.\end{aligned}$$

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