Problem 12303
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Let R and r be the circumradius and inradius, respectively, of triangle ABC. Let D, E, and F be chosen on sides BC, CA, and AB so that AD, BE, and CF bisect the angles of ABC. Prove

$$\frac{FD}{AB+BC}+\frac{DE}{BC+CA}+\frac{EF}{CA+AB}\leq\frac{3}{8}\left(1+\frac{R}{2r}\right).$$

Solution proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.

Solution. Let $a=BC$, $b=CA$ and $c=AB$. Let S and s be the area and semiperimeter of ABC, respectively. By the angle bisector theorem, $AE=\frac{bc}{a+c}$ and $AF=\frac{bc}{a+b}$. Then by the law of cosines,

$$\begin{aligned}EF&=\sqrt{AE^2+AF^2-2AE\cdot AF\cos(A)}\\&=\frac{bc}{(a+b)(a+c)}\sqrt{(a+b)^2+(a+c)^2-2(a+b)(a+c)\cos(A)}\\&=\frac{abc\sqrt{3+2(\cos(B)+\cos(C)-\cos(A))}}{(a+b)(a+c)}.\end{aligned}$$

Similar formulas hold for the sides FD and DE. Hence,

$$\begin{aligned}\frac{FD}{AB+BC}+\frac{DE}{BC+CA}+\frac{EF}{CA+AB}&=\frac{abc}{(a+b)(b+c)(c+a)}\sum_{\text{cyc}}\sqrt{3+2(\cos(B)+\cos(C)-\cos(A))}\\&\leq\frac{3}{8}\sqrt{\frac{1}{3}\sum_{\text{cyc}}(3+2(\cos(B)+\cos(C)-\cos(A)))}\\&=\frac{3}{8}\sqrt{3+\frac{2}{3}(\cos(A)+\cos(B)+\cos(C))}\\&=\frac{3}{8}\sqrt{3+\frac{2}{3}\left(1+\frac{r}{R}\right)}=\frac{3}{8}\sqrt{\frac{11}{3}+\frac{2r}{3R}}\end{aligned}$$

where we applied the inequaly $8abc\leq(a+b)(b+c)(c+a)$ (because $\frac{x+y}{2}\geq\sqrt{xy}$), the concavity of the function $t\to\sqrt{t}$, and the identity (recall that $S=rs=\frac{abc}{4R}$),

$$\begin{aligned}\cos(A)+\cos(B)+\cos(C)-1&=\frac{b^2+c^2-a^2}{2bc}+\frac{c^2+a^2-b^2}{2ca}+\frac{a^2+b^2-c^2}{2ab}-1\\&=\frac{(a+b-c)(b+c-a)(c+a-b)}{2abc}=\frac{8S^2}{2abcs}=\frac{r}{R}.\end{aligned}$$

Therefore, it remains to show that for any $t=\frac{2r}{R}\in(0,1]$ ($2r\leq R$ by Euler's theorem),

$$\sqrt{\frac{11}{3}+\frac{t}{3}}\leq 1+\frac{1}{t}$$

that is

$$t^2(11+t)\leq 3(t+1)^2\Leftrightarrow t^3+8t^2-6t-3\leq 0\Leftrightarrow(t-1)(t^2+9t+3)\leq 0$$

which holds and we are done. □