Problem 12302
(American Mathematical Monthly, Vol.129, February 2022)

Proposed by M. Omarjee (France).

Let n be a positive integer, and let A be the $2n$-by-$2n$ skew-symmetric matrix with (j,k)-entry $\sin(j-k)/\sin(j+k)$. Prove

$$\det(A) = \prod_{1\leq j<k\leq 2n} \left(\frac{\sin(j-k)}{\sin(j+k)}\right)^2 .$$

Solution proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.

Solution. We will show that

$$\det\left(M_n\right) = \prod_{1\leq j<k\leq 2n} \left(\frac{x_j - x_k}{x_j + x_k}\right)^2 \tag{1}$$

where M_n is $2n$-by-$2n$ skew-symmetric matrix with (j,k)-entry $f_{j,k} = \frac{x_j - x_k}{x_j + x_k}$.
Then the result follows by letting $x_j = \tan(j)$:

$$\frac{\sin(j-k)}{\sin(j+k)} = \frac{\sin(j)\cos(k) - \cos(j)\sin(k)}{\sin(j)\cos(k) - \cos(j)\sin(k)} = \frac{\tan(j) - \tan(k)}{\tan(j) + \tan(k)} = \frac{x_j - x_k}{x_j + x_k}.$$

We prove (1) by induction with respect to n.
It is trivial for $n = 1$. As regards the induction step, we notice that

$$M_{n+2} = \left[\begin{array}{ccc|cc} f_{1,1} & \cdots & f_{1,n} & f_{1,n+1} & f_{1,n+2} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ f_{n,1} & \cdots & f_{n,n} & f_{n,n+1} & f_{n,n+2} \\ \hline -f_{1,n+1} & \cdots & -f_{n,n+1} & 0 & f_{n+1,n+2} \\ -f_{1,n+2} & \cdots & -f_{n,n+2} & -f_{n+1,n+2} & 0 \end{array}\right]$$

has the same determinant as the matrix

$$\left[\begin{array}{ccc|cc} g_{1,1} & \cdots & g_{1,n} & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ g_{n,1} & \cdots & g_{n,n} & 0 & 0 \\ \hline -f_{1,n+1} & \cdots & -f_{n,n+1} & 0 & f_{n+1,n+2} \\ -f_{1,n+2} & \cdots & -f_{n,n+2} & -f_{n+1,n+2} & 0 \end{array}\right]$$

where we substracted from the first n rows a suitable multiple of the last two rows in order to get zero in the last two entries. It can be verified that

$$g_{j,k} = f_{j,k} + \frac{f_{j,n+2}f_{k,n+1} - f_{j,n+1}f_{k,n+2}}{f_{n+1,n+2}} = f_{j,k}f_{j,n+1}f_{j,n+2}f_{k,n+1}f_{k,n+2},$$

hence we may conclude

$$\det(M_{n+2}) = \det([g_{j,k}]) \cdot f_{n+1,n+2}^2 = \det(M_n) \cdot \prod_{j=1}^{n}(f_{j,n+1}f_{j,n+2}) \cdot \prod_{k=1}^{n}(f_{k,n+1}f_{k,n+2}) \cdot f_{n+1,n+2}^2$$

$$= \prod_{1\leq j<k\leq 2n} f_{j,k}^2 \cdot \prod_{j=1}^{n}(f_{j,n+1}f_{j,n+2})^2 \cdot f_{n+1,n+2}^2 = \prod_{1\leq j<k\leq 2n+2} f_{j,k}^2$$

and we are done. $\square$