Problem 12299
(American Mathematical Monthly, Vol.129, January 2022)

Proposed by E. Vigren (Sweden).

Let $x_{0,n} = x_{1,n} = 1$ and, for integers k with $2 \le k \le n-1$, let

$$x_{k,n} = \frac{1}{k}\Big(nx_{k-1,n} - \sum_{j=1}^{k-1} x_{j,n}\Big).$$

Let $T_n = n^2 x_{n-1,n} - n + 1$. The first few values of T_n are

$$1, 3, 7, 47/3, 427/12, 416/5.$$

Prove that T_n is the expected number of throws of an n-sided die until the last n throws contain all possible face values.

Solution proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.

Solution. Given, $n \ge 3$, it is straightforward to verify by induction that for $k \ge 1$,

$$x_{k,n} = \frac{1}{k!}\Big(n^{k-1} - \sum_{j=0}^{k-2}(k-1-j)j!n^{k-2-j}\Big)$$

and it follows that

$$\begin{aligned}
T_n = n^2 x_{n-1,n} - n + 1 &= \frac{1}{(n-1)!}\Big(n^n - \sum_{j=0}^{n-3}(n-2-j)j!n^{n-1-j}\Big) - n + 1\\
&= \frac{1}{(n-1)!}\Big(n^n + \sum_{j=0}^{n-3} j!n^{n-1-j} + \sum_{j=0}^{n-3}((j+1)!n^{n-(j+1)} - j!n^{n-j})\Big) - n + 1\\
&= \frac{1}{(n-1)!}\Big(n^n + \sum_{j=0}^{n-3} j!n^{n-1-j} + (n-2)!n^2 - n^n\Big) - n + 1\\
&= \frac{1}{n!}\sum_{j=0}^{n-3} j!n^{n-j} + \frac{n^2}{n-1} - n + 1 = \frac{1}{n!}\sum_{j=0}^{n-1} j!n^{n-j}.
\end{aligned}$$

We consider the random process of throwing the n-sided die as Markov Chain with states $s_1, \dots, s_n$. The state s_j is the set of all sequences of throws where the last j values are all distinct, but the last $j+1$ values are not. Let $E(s_j)$ be the expected number of throws to reach state s_n, starting from s_j. Then trivially $E(s_n) = 0$. Furthermore, for $1 \le j \le n-1$,

$$E(s_j) = \frac{1}{n}\sum_{i=1}^{j}(E(s_i)+1) + \Big(1 - \frac{j}{n}\Big)(E(s_{j+1})+1) = \frac{1}{n}\sum_{i=1}^{j} E(s_i) + \Big(1 - \frac{j}{n}\Big)E(s_{j+1}) + 1 \qquad (1)$$

because from state s_j there is probability $1/n$ to go to any state $s_1, \dots, s_j$, and there is probability $1 - j/n$ to go to state s_{j+1}. Hence it remains to verify that $T_n = E(s_1) + 1$ (at the beginning we need one extra-step to reach state s_1).
By subtraction, for $2 \le j \le n-1$,

$$E(s_{j-1}) - E(s_j) = \Big(1 - \frac{j-1}{n}\Big)E(s_j) - \frac{E(s_j)}{n} - \Big(1 - \frac{j}{n}\Big)E(s_{j+1}) = \frac{n-j}{n}(E(s_j) - E(s_{j+1}))$$

which implies that

$$\begin{aligned}
E(s_{j-1}) - E(s_j) &= \frac{n-j}{n}(E(s_j) - E(s_{j+1})) = \frac{(n-j)(n-j-1)}{n^2}(E(s_{j+1}) - E(s_{j+2}))\\
&= \cdots = \frac{(n-j)!}{n^{n-j}}(E(s_{n-1}) - E(s_n)) = \frac{(n-j)!}{n^{n-j}}E(s_{n-1}).
\end{aligned}$$

Therefore, for $1 \le i \le n-2$,

$$E(s_i) = \sum_{j=i+1}^{n-1} (E(s_{j-1}) - E(s_j)) + E(s_{n-1}) = E(s_{n-1}) \sum_{j=i+1}^{n} \frac{(n-j)!}{n^{n-j}} = E(s_{n-1}) \sum_{j=0}^{n-i-1} \frac{j!}{n^j}.$$

Letting $j = n-1$ in (1), we get

$$\begin{aligned} E(s_{n-1}) &= \frac{1}{n} \sum_{i=1}^{n-2} E(s_i) + \frac{E(s_{n-1})}{n} + 1 = \frac{E(s_{n-1})}{n} \sum_{i=1}^{n-1} \sum_{j=0}^{n-i-1} \frac{j!}{n^j} + 1 \\ &= E(s_{n-1}) \sum_{j=0}^{n-2} \frac{j!(n-j-1)}{n^{j+1}} + 1 = E(s_{n-1}) \sum_{j=0}^{n-2} \Big(\frac{j!}{n^j} - \frac{(j+1)!}{n^{j+1}} \Big) + 1 \\ &= E(s_{n-1}) \Big(1 - \frac{n!}{n^n} \Big) + 1 \end{aligned}$$

and we find that $E(s_{n-1}) = n^n/n!$. Finally,

$$E(s_1) + 1 = E(s_{n-1}) \sum_{j=0}^{n-2} \frac{j!}{n^j} + 1 = \frac{1}{n!} \sum_{j=0}^{n-2} j! n^{n-j} + 1 = \frac{1}{n!} \sum_{j=0}^{n-1} j! n^{n-j}$$

which is the same formula given above for T_n, and we are done. □