Problem 12298
(American Mathematical Monthly, Vol.129, January 2022)

Proposed by G. Stoica (Canada).

Let n be a positive integer, S_n be the group of all permutations of $\{1, 2, \ldots, n\}$, and z be a primitive complex nth root of unity. Prove

$$\sum_{\sigma \in S_n} \prod_{j=1}^{n} \left(1 - x_j z^{\sigma(j)}\right) = n! \left(1 - \prod_{i=1}^{n} x_i\right)$$

for any $x_1, \ldots, x_n \in \mathbb{C}$.

Solution proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.

Solution. We first note that if $J \subseteq \{1, \ldots, n\}$ with $|J| = k$, and $z \in \mathbb{C}$ then, by the q-binomial theorem,

$$\sum_{\sigma \in S_n} \prod_{j \in J} z^{\sigma(j)} = (n-k)!k! \cdot [t^k] \prod_{i=1}^{n} (1 + tz^j) = (n-k)!k! \cdot z^{k(k+1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_z$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix}_z = \frac{(1-z^n)(1-z^{n-1}) \cdots (1-z^{n-k+1})}{(1-z^k) \cdots (1-z)}$$

are the Gaussian binomial coefficients (which are polynomials with respect to z).
Therefore, for any complex number z,

$$\begin{aligned} P_n(z) = \sum_{\sigma \in S_n} \prod_{j=1}^{n} \left(1 - x_j z^{\sigma(j)}\right) &= \sum_{\sigma \in S_n} \sum_{k=0}^{n} (-1)^k \sum_{\substack{J \subseteq \{1,\ldots,n\} \\ |J|=k}} \prod_{j \in J} x_j z^{\sigma(j)} \\ &= \sum_{k=0}^{n} (-1)^k \sum_{\substack{J \subseteq \{1,\ldots,n\} \\ |J|=k}} \prod_{j \in J} x_j \cdot \sum_{\sigma \in S_n} \prod_{j \in J} z^{\sigma(j)} \\ &= \sum_{k=0}^{n} (-1)^k (n-k)!k! \cdot z^{k(k+1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_z \cdot \sum_{\substack{J \subseteq \{1,\ldots,n\} \\ |J|=k}} \prod_{j \in J} x_j. \end{aligned}$$

Now, if z is a primitive complex nth root of unity then $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_z = 1$ when $k \in \{0, n\}$ and it is zero otherwise, and it follows that

$$\begin{aligned} P_n(z) &= \sum_{k \in \{0,n\}} (-1)^k (n-k)!k! \cdot z^{k(k+1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_z \cdot \sum_{\substack{J \subseteq \{1,\ldots,n\} \\ |J|=k}} \prod_{j \in J} x_j \\ &= n! + (-1)^n n! \cdot z^{n(n+1)/2} \cdot \prod_{i=1}^{n} x_i = n! \left(1 - \prod_{i=1}^{n} x_i\right). \end{aligned}$$

$\square$