Problem 12297
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Proposed by N. Bhandari (Nepal).

Prove

$$\int_0^{\pi/2} \left(\frac{\sinh^{-1}(\sin(x))}{\sin(x)}\right)^2 dx = \frac{\pi}{2}\left(\frac{\pi}{2} - \ln(2)\right).$$

Solution proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.

Solution. It is known that for $|z| \leq 1$,

$$\left(\sinh^{-1}(z)\right)^2 = \frac{1}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}(2x)^{2k}}{k^2\binom{2k}{k}}.$$

Therefore

$$\begin{aligned}
\int_0^{\pi/2} \left(\frac{\sinh^{-1}(\sin(x))}{\sin(x)}\right)^2 dx &= \frac{1}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}4^k}{k^2\binom{2k}{k}} \int_0^{\pi/2} \sin^{2k-2}(x)\, dx \\
&= \pi \sum_{k=1}^{\infty} \frac{(-1)^{k-1}4^k}{4k^2\binom{2k}{k}} \cdot \frac{\binom{2k-2}{k-1}}{4^{k-1}} \\
&= \pi \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k(2k-1)} \\
&= \pi \sum_{k=1}^{\infty} (-1)^{k-1}\left(\frac{1}{2k-1} - \frac{1}{2k}\right) \\
&= \pi \left(\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k-1} - \frac{1}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}\right) \\
&= \pi\left(\frac{\pi}{4} - \frac{1}{2}\ln(2)\right) = \frac{\pi}{2}\left(\frac{\pi}{2} - \ln(2)\right)
\end{aligned}$$

where we used Wallis' integral formula

$$\int_0^{\pi/2} \sin^{2n}(x)\, dx = \frac{\pi}{2} \cdot \frac{\binom{2n}{n}}{4^n}.$$

□