Problem 12296
(American Mathematical Monthly, Vol.129, January 2022)

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For $t \leq n/2$, let $H(n,t)$ be the graph obtained from the complete graph on n vertices by deleting t pairwise disjoint edges. Determine the number of ways to assign each vertex of $H(n,t)$ a color from a set of k available colors so that vertices forming an edge receive distinct colors.

Solution proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.

Solution. We have to find the so-called *chromatic polynomial* $P_G(k)$ of the graph $G = H(n,t)$.
We will show that

$$P_{H(n,t)}(k) = \sum_{j=0}^{t} \binom{t}{j} P_{H(n-j,0)}(k) = \sum_{j=0}^{t} \binom{t}{j} k(k-1)\dots(k-(n-j)+1)$$

where $P_{H(m,0)}(k) = k(k-1)\dots(k-m+1)$ is the chromatic polynomial of the complete graph on m vertices $H(m,0)$.

The proof is by double induction with respect to (n,t) with $n \geq 1$ and $0 \leq t \leq n/2$. The claim is trivially true for all $n \geq 1$ and $t = 0$.
For the inductive step, we assume $t > 1$ and we recall the following fundamental property of the chromatic polynomials: if G is a simple graph, then

$$P_{G-e}(k) = P_G(k) + P_{G/e}(k)$$

where $G - e$ and G/e are, respectively, the graphs obtained from G by deleting and contracting an edge e. Therefore, by letting $G = H(n,t-1)$, we obtain

$$\begin{aligned}
P_{H(n,t)}(k) &= P_{H(n,t-1)}(k) + P_{H(n-1,t-1)}(k)\\
&= \sum_{j=0}^{t-1} \binom{t-1}{j} P_{H(n-j,0)}(k) + \sum_{j=0}^{t-1} \binom{t-1}{j} P_{H(n-1-j,0)}(k)\\
&= \sum_{j=0}^{t-1} \binom{t-1}{j} P_{H(n-j,0)}(k) + \sum_{j=1}^{t} \binom{t-1}{j-1} P_{H(n-j,0)}(k)\\
&= P_{H(n,0)}(k) + \sum_{j=1}^{t-1} \left(\binom{t-1}{j} + \binom{t-1}{j-1} \right) P_{H(n-j,0)}(k) + P_{H(n,t)}(k)\\
&= \sum_{j=0}^{t} \binom{t}{j} P_{H(n-j,0)}(k)
\end{aligned}$$

and the proof is complete. $\square$