Problem 12294
(American Mathematical Monthly, Vol.129, January 2022)

Proposed by Tran Quang Hung (Vietnam).

Let $A_1A_2A_3A_4$ be a quadrilateral inscribed in a circle with center O. Let $B_1B_2B_3B_4$ be the quadrilateral that contains $A_1A_2A_3A_4$ in its interior such that, for $1 \leq k \leq 4$ and with subscripts taken cyclically, B_kB_{k+1} is parallel to A_kA_{k+1} and at distance $|A_kA_{k+1}|$ from it. Because $B_1B_2B_3B_4$ has the same angles as $A_1A_2A_3A_4$, there is a circle in which it is inscribed. Let P be the center of that circle. Show that A_1A_3, A_2A_4, and OP are concurrent.

Solution proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.

Solution. We will show more general result: the concurrency holds when B_kB_{k+1} is at distance $r|A_kA_{k+1}|$ from A_kA_{k+1} for $1 \leq k \leq 4$ and for any $r > 0$.

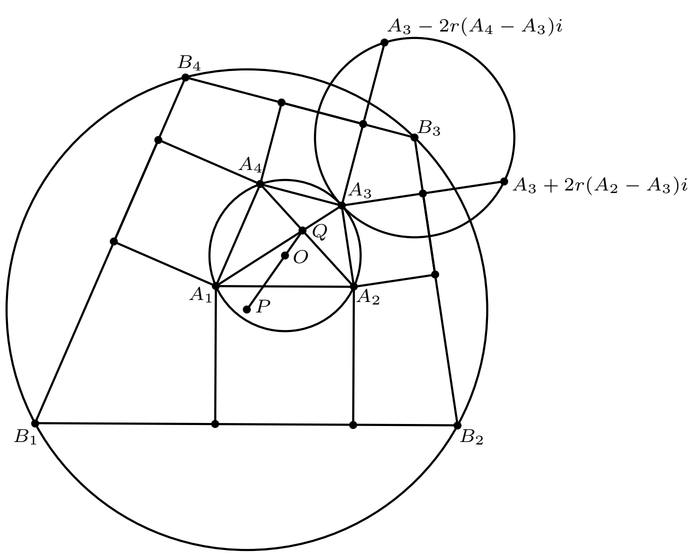
In the complex plane, without loss of generality, we may assume that the circumcircle of $A_1A_2A_3A_4$ is the unit circle $|z| = 1$ and

$$A_1 = \frac{1-a^2}{1+a^2} + \frac{2ai}{1+a^2},\ A_2 = \frac{1-b^2}{1+b^2} + \frac{2bi}{1+b^2},\ A_3 = \frac{1-c^2}{1+c^2} + \frac{2ci}{1+c^2},\ A_4 = \frac{1-d^2}{1+d^2} + \frac{2di}{1+d^2}$$

with $a < b < c < d$ real numbers.
The points A_1, A_2, A_3, A_4 lie on the unit circle and therefore the intersection of A_1A_3 and A_2A_4 is

$$Q := \frac{A_1A_3(A_2 + A_4) - A_2A_4(A_1 + A_3)}{A_1A_3 - A_2A_4} = \frac{2(a + c - b - d) + 2(ac - bd)i}{(ab + 1)(c - d) + (cd + 1)(a - b)} - 1.$$

We note that the point B_3 is the circumcenter of the triangle of vertices A_3, $A_3 + 2r(A_2 - A_3)i$, and $A_3 - 2r(A_4 - A_3)i$ and a similar property holds for B_1, B_2, B_4.



Hence we find that

$$B_1 = A_1 + 2ri\,C(0, A_1 - A_2, A_4 - A_1),$$
$$B_2 = A_2 + 2ri\,C(0, A_2 - A_3, A_1 - A_2),$$
$$B_3 = A_3 + 2ri\,C(0, A_3 - A_4, A_2 - A_3),$$
$$B_4 = A_4 + 2ri\,C(0, A_4 - A_1, A_3 - A_4),$$

where the circumcenter of the triangle with vertices $u, v, w \in \mathbb{C}$ is given by the ratio

$$C(u,v,w) = \begin{vmatrix} u & |u|^2 & 1 \\ v & |v|^2 & 1 \\ w & |w|^2 & 1 \end{vmatrix} \div \begin{vmatrix} u & \overline{u} & 1 \\ v & \overline{v} & 1 \\ w & \overline{w} & 1 \end{vmatrix}.$$

Then, it can be verified that

$$\begin{aligned} P = C(B_1, B_2, B_3) &= 2r \frac{2(a+c-b-d) - (ab+1)(c-d) - (cd+1)(a-b) + 2(ac-bd)i}{(a-c)(b-d)} \\ &= 2r \frac{(ab+1)(c-d) + (cd+1)(a-b)}{(a-c)(b-d)} \cdot Q. \end{aligned}$$

Since $O = 0$ and $2r \frac{(ab+1)(c-d)+(cd+1)(a-b)}{(a-c)(b-d)}$ is a real number, it follows that the points Q, O, and P are collinear, that is A_1A_3, A_2A_4, and OP are concurrent. $\square$