Problem 12293
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For any integer $n \geq 1$, and any real number $r > 0$, prove

$$\sum_{k=0}^{n}(-1)^k\left(\sum_{j=0}^{k} r^j\binom{n}{j}\right)\left(\sum_{j=0}^{k}(-r)^j\binom{n}{j}\right) = \left(\frac{(r+1)^n+(r-1)^n}{2}\right)^2.$$

Solution proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.

Solution. It suffices to show that for any real numbers $a_0, a_1, \ldots, a_n$,

$$\sum_{k=0}^{n}(-1)^k\left(\sum_{j=0}^{k} a_j\right)\left(\sum_{j=0}^{k}(-1)^j a_j\right) = \left(\sum_{s=0}^{\lfloor n/2\rfloor} a_{n-2s}\right)^2. \tag{1}$$

Then, by letting $a_j = r^j\binom{n}{j}$ we have that

$$\sum_{s=0}^{\lfloor n/2\rfloor} a_{n-2s} = \sum_{s=0}^{\lfloor n/2\rfloor}\binom{n}{n-2s}r^{n-2s} = \frac{(r+1)^n+(r-1)^n}{2}.$$

We show (1) when n is even (otherwise the proof is similar).
If $0 \leq s \leq n$ then the coefficient of a_s^2 on the left side of (1) is

$$(-1)^s\sum_{k=s}^{n}(-1)^k = \begin{cases}1 & \text{if } s \equiv 0 \pmod 2\\ 0 & \text{otherwise}\end{cases}.$$

If $0 \leq s < t \leq n$ then the coefficient of $a_s a_t$ on the left side of (1) is

$$((-1)^s+(-1)^t)\sum_{k=t}^{n}(-1)^k = \begin{cases}2 & \text{if } s \equiv t \equiv 0 \pmod 2\\ 0 & \text{otherwise}\end{cases}.$$

Hence the left side of (1) is equal to

$$\sum_{s=0}^{n/2} a_{2s}^2 + 2\sum_{s=0}^{n/2}\sum_{t=s+1}^{n/2} a_{2s}a_{2t} = \left(\sum_{s=0}^{n/2} a_{2s}\right)^2 = \left(\sum_{s=0}^{n/2} a_{n-2s}\right)^2$$

and we are done. □