**Problem 12292**
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*Let $p$ be a prime number, and let $r = 1/(2\cos(4\pi/7))$. Evaluate $\lfloor r^{p+2} \rfloor$ modulo $p$.*

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*Solution.* We show that for any prime $p$,

$$\lfloor (2\cos(4\pi/7))^{-(p+2)} \rfloor \equiv \begin{cases} 2 & \text{if } p \equiv 0,3,4 \pmod 7 \\ -5 & \text{if } p \equiv 2,5 \pmod 7 \\ -12 & \text{if } p \equiv 1,6 \pmod 7 \end{cases} \pmod p. \tag{1}$$

We divide the proof in several steps.

**1)** For $n \geq 0$, let $C_n = (2\cos(2\pi/7))^n + (2\cos(4\pi/7))^n + (2\cos(6\pi/7))^n$, then

$$C_{2n} = \frac{7}{2} \sum_{\substack{k=0 \\ k \equiv n \,(\mathrm{mod}\ 7)}}^{2n} \binom{2n}{k} - 2^{2n-1} \quad \text{and} \quad C_{2n+1} = 7 \sum_{\substack{k=0 \\ k \equiv n+3 \,(\mathrm{mod}\ 7)}}^{2n} \binom{2n}{k} - 2^{2n}.$$

Let $f(z) = \sum_{k \geq 0} a_k z^k$ and $\omega_m = e^{\frac{2\pi i}{m}}$ then

$$\sum_{\substack{k \geq 0 \\ k \equiv d \,(\mathrm{mod}\ m)}} a_k z^k = \sum_{k \geq 0} \left( \frac{1}{m} \sum_{j=0}^{m-1} \omega_m^{(d-k)j} \right) a_k z^k = \frac{1}{m} \sum_{j=0}^{m-1} \omega_m^{dj} f(\omega_m^{-j} z).$$

If $f(z) = (1+z)^{2n}$ then $a_k = \binom{2n}{k}$ and

$$\sum_{\substack{k=0 \\ k \equiv d \,(\mathrm{mod}\ m)}}^{2n} \binom{2n}{k} = \frac{1}{m} \sum_{j=0}^{m-1} \omega_m^{dj} (1 + \omega_m^{-j})^{2n} = \frac{2^{2n}}{m} + \frac{1}{m} \sum_{j=1}^{m-1} \omega_m^{(d-n)j} (\omega_m^{\frac{j}{2}} + \omega_m^{-\frac{j}{2}})^{2n}$$

$$= \frac{2^{2n}}{m} + \frac{1}{m} \sum_{j=1}^{\frac{m-1}{2}} 2\cos\left( \frac{\pi(2d-2n)j}{m} \right) \left( 2\cos\left( \frac{\pi j}{m} \right) \right)^{2n}.$$

The above formulas follow by letting $m = 7$, $d = n$, and $d = n+3$.

**2)** For $n \geq 0$, let $C_{-n} = (2\cos(2\pi/7))^{-n} + (2\cos(4\pi/7))^{-n} + (2\cos(6\pi/7))^{-n}$ then

$$C_{-n} = \frac{C_n^2 - C_{2n}}{2}.$$

Let $\alpha = 2\cos(2\pi/7)$, $\beta = 2\cos(4\pi/7)$, $\gamma = 2\cos(6\pi/7)$. Since $\alpha\beta\gamma = 1$ then

$$2C_{-n} = 2(\beta\gamma)^n + 2(\alpha\gamma)^n + 2(\alpha\beta)^n = (\alpha^n + \beta^n + \gamma^n)^2 - (\alpha^{2n} + \beta^{2n} + \gamma^{2n}) = C_n^2 - C_{2n}.$$

**3)** For $n \geq 2$,

$$C_{-n} = \lfloor (2\cos(4\pi/7))^{-n} \rfloor + 1.$$

Note that $|(2\cos(4\pi/7))^{-1}| > 1$, $-0.56 < (2\cos(6\pi/7))^{-1} < 0 < (2\cos(2\pi/7))^{-1} < 0.81$. Then the claim follows because for $n \geq 2$,

$$0 < (2\cos(2\pi/7))^{-n} + (2\cos(6\pi/7))^{-n} < (0.81)^2 + (0.56)^2 < 1.$$

**4)** For any prime $p \neq 2, 7$,

$$C_{p+2} \equiv \begin{cases} 3 & \text{if } p \equiv 2,5 \pmod 7 \\ -4 & \text{if } p \equiv 1,3,4,6 \pmod 7 \end{cases} \pmod p.$$

By Lucas Theorem, $\binom{p+1}{k} \equiv \binom{1}{k_1}\binom{1}{k_0}$ modulo $p$ where $k = pk_1 + k_0$ in base $p$, and therefore

$$S := \sum_{\substack{k=0 \\ k \equiv \frac{p+1}{2}+3 \pmod 7}}^{p+1} \binom{p+1}{k} \equiv \sum_{\substack{k=0 \\ 2k \equiv p \pmod 7}}^{1} 1 + \sum_{\substack{k=p \\ 2k \equiv p \pmod 7}}^{p+1} 1 \equiv \begin{cases} 1 & \text{if } p \equiv 2,5 \pmod 7 \\ 0 & \text{if } p \equiv 1,3,4,6 \pmod 7 \end{cases} \pmod p.$$

Hence, since $p+2$ is odd, it follows that $C_{p+2} = 7S - 2^{p+1} \equiv 7S - 4 \pmod p$, and we are done.

**5)** For any prime $p \neq 2, 7$,

$$C_{2p+4} \equiv \begin{cases} 10 & \text{if } p \equiv 3,4 \pmod 7 \\ 17 & \text{if } p \equiv 2,5 \pmod 7 \\ 38 & \text{if } p \equiv 1,6 \pmod 7 \end{cases} \pmod p.$$

By Lucas Theorem, $\binom{2p+4}{k} \equiv \binom{2}{k_1}\binom{4}{k_0}$ modulo $p$, and therefore

$$S := \sum_{\substack{k=0 \\ k \equiv p+2 \pmod 7}}^{2p+4} \binom{2p+4}{k} \equiv 2\binom{4}{2} + 2 \sum_{\substack{k=0 \\ k \equiv p+2 \pmod 7}}^{4} \binom{4}{k} \equiv \begin{cases} 12 & \text{if } p \equiv 3,4 \pmod 7 \\ 14 & \text{if } p \equiv 2,5 \pmod 7 \\ 20 & \text{if } p \equiv 1,6 \pmod 7 \end{cases} \pmod p.$$

Hence, since $2p+4$ is even, it follows that $C_{2p+4} = \frac{7}{2}S - 2^{2p+3} \equiv \frac{7}{2}S - 32 \pmod p$ and we are done.

**6)** We directly check that (1) holds for $p = 2, 7$. For any other prime $p$, by the previous steps, we may conclude that

$$\begin{aligned} \lfloor (2\cos(4\pi/7))^{-(p+2)} \rfloor &= C_{-(p+2)} - 1 = \frac{C_{p+2}^2 - C_{2p+4}}{2} - 1 \\ &\equiv \begin{cases} \frac{(-4)^2-10}{2} - 1 & \text{if } p \equiv 3,4 \pmod 7 \\ \frac{3^2-17}{2} - 1 & \text{if } p \equiv 2,5 \pmod 7 \\ \frac{(-4)^2-38}{2} - 1 & \text{if } p \equiv 1,6 \pmod 7 \end{cases} \pmod p. \end{aligned}$$

and the formula (1) is fully verified. □

**Remarks.** $\{C_n\}_{n \geq 0}$ is the OEIS sequence A094648:

$$3, -1, 5, -4, 13, -16, 38, -57, 117, -193, 370, -639, 1186, -2094, 3827, \ldots$$

and it satisfies the recurrence formula $C_{n+3} = -C_{n+2} + 2C_{n+1} + C_n$. By using a similar approach we show that for any prime $p$,

$$C_p \equiv -1 \pmod p, \quad C_{2p} \equiv 5 \pmod p, \quad \text{and} \quad \lfloor (2\cos(4\pi/7))^{-p} \rfloor \equiv -3 \pmod p.$$