Problem 12921
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The Nagel point of a triangle is the point common to the three segments that join a vertex of the triangle to the point at which an excircle touches the opposite side. Let ABC be a triangle with incenter I and Nagel point J. Prove that AJ is perpendicular to the line through the orthocenters of triangles IAB and IAC.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We solve the problem by using barycentric coordinates: each point P in the plane is assigned an ordered triple of real numbers (x, y, z) such that $x+y+z=1$ and $\vec{P} = x\vec{A} + y\vec{B} + z\vec{C}$. Then $A = (1, 0, 0)$, $B = (0, 1, 0)$, $C = (0, 0, 1)$, and it is known that

$$I = \left(\frac{a}{2s}, \frac{b}{2s}, \frac{c}{2s}\right), \quad J = \left(\frac{s-a}{s}, \frac{s-b}{s}, \frac{s-c}{s}\right)$$

where $a = |BC|$, $b = |CA|$, $c = |AB|$ and $s = (a+b+c)/2$.
Moreover, the orthocenter H_B of the triangle IAC has (not normalized) barycentric coordinates with respect to IAC:

$$\left(\frac{1}{b'^2+c'^2-a'^2}, \frac{1}{c'^2+a'^2-b'^2}, \frac{1}{a'^2+b'^2-c'^2}\right)$$

where $a' = |AC| = b$, $b' = |IC| = \sqrt{\frac{ab(a+b-c)}{a+b+c}}$, $c' = |IA| = \sqrt{\frac{bc(b+c-a)}{a+b+c}}$.
We recall that:
Let (p, q, r) be the barycentric coordinates of a point P with respect to a triangle DEF, such that $D = (u_1, v_1, w_1)$, $E = (u_2, v_2, w_2)$, $F = (u_3, v_3, w_3)$ are the barycentric coordinates with respect to the triangle ABC. Then the barycentric coordinates of P with respect to triangle ABC are as follows

$$(u_1p + u_2q + u_3r, v_1p + v_2q + v_3r, w_1p + w_2q + w_3r).$$

Therefore we find the barycentric coordinates of H_B with respect to the triangle ABC:

$$H_B = \left(\frac{c-b}{a+c-b}, \frac{b}{a+c-b}, \frac{a-b}{a+c-b}\right).$$

In a similar way, the orthocenter H_C of the triangle IAB has barycentric coordinates

$$H_C = \left(\frac{b-c}{a+b-c}, \frac{a-c}{a+b-c}, \frac{c}{a+b-c}\right).$$

Finally, we show that AJ is perpendicular to H_BH_C: we verify that

$$(\vec{A} - \vec{J}) \cdot (\vec{H}_B - \vec{H}_C) = 0$$

which is a tedious, but fairly straightforward task. Just notice that, after translating the circumcenter of ABC to the origin, we have

$$\vec{A} \cdot \vec{A} = \vec{B} \cdot \vec{B} = \vec{C} \cdot \vec{C} = R^2, \quad \vec{A} \cdot \vec{B} = R^2 \cos(2C) = R^2 - \frac{c^2}{2}, \ \vec{B} \cdot \vec{C} = R^2 - \frac{a^2}{2}, \ \vec{C} \cdot \vec{A} = R^2 - \frac{b^2}{2}$$

where R is the circumradius (we may assume $R = 1$ without loss of generality).

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