Problem 12289
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Prove
$$\sum_{n=0}^{\infty} 2\cos\left(\frac{(2n+1)\pi}{3}\right) q^{n(n+1)/2} = \prod_{n=1}^{\infty}(1-q^n)(1-q^{6n-1})(1-q^{6n-5})$$
when $|q|<1$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let $z = e^{\pi i/3}$, then

$$\begin{aligned}
z\sum_{n=-\infty}^{\infty} z^{2n}q^{n(n+1)/2} &= \sum_{n=1}^{\infty} z^{-2n+1}q^{n(n-1)/2} + \sum_{n=0}^{\infty} z^{2n+1}q^{n(n+1)/2}\\
&= \sum_{n=0}^{\infty} z^{-2(n+1)+1}q^{n(n+1)/2} + \sum_{n=0}^{\infty} z^{2n+1}q^{n(n+1)/2}\\
&= \sum_{n=0}^{\infty}(z^{2n+1}+z^{-(2n+1)})q^{n(n+1)/2} = \sum_{n=0}^{\infty} 2\cos\left(\frac{(2n+1)\pi}{3}\right) q^{n(n+1)/2}.
\end{aligned}$$

On the other hand, by the Jacobi's triple product identity,

$$\sum_{n=-\infty}^{\infty} z^{2n}q^{n(n+1)/2} = \prod_{n=1}^{\infty}(1-q^n)(1+z^2q^n)(1+z^{-2}q^{n-1}).$$

Therefore, since $z^3=-1$, it follows that

$$\begin{aligned}
z\sum_{n=-\infty}^{\infty} z^{2n}q^{n(n+1)/2} &= z\prod_{n=1}^{\infty}(1-q^n)(1+z^2q^n)(1+z^{-2}q^{n-1})\\
&= z(1+z^{-2})\prod_{n=1}^{\infty}(1-q^n)(1+z^2q^n)(1+z^{-2}q^n)\\
&= \prod_{n=1}^{\infty}(1-q^n)\prod_{n=1}^{\infty}(z-q^n)(z^{-1}-q^n)\\
&= \prod_{n=1}^{\infty}(1-q^n)\prod_{n=1}^{\infty}\frac{1+q^{3n}}{1+q^n}\\
&= \prod_{n=1}^{\infty}(1-q^n)\prod_{n=1}^{\infty}\frac{1-q^{6n}}{1-q^{3n}}\prod_{n=1}^{\infty}\frac{1-q^n}{1-q^{2n}}\\
&= \prod_{n=1}^{\infty}(1-q^n)\frac{\prod_{n=1}^{\infty}(1-q^{6n})\prod_{n=1}^{\infty}(1-q^n)}{\prod_{n=1}^{\infty}(1-q^{6n})(1-q^{6n-3})\prod_{n=1}^{\infty}(1-q^{6n})(1-q^{6n-2})(1-q^{6n-4})}\\
&= \prod_{n=1}^{\infty}(1-q^n)\frac{\prod_{n=1}^{\infty}(1-q^n)}{\prod_{n=1}^{\infty}(1-q^{6n})(1-q^{6n-2})(1-q^{6n-3})(1-q^{6n-4})}\\
&= \prod_{n=1}^{\infty}(1-q^n)(1-q^{6n-1})(1-q^{6n-5}).
\end{aligned}$$

□