Problem 12288
(American Mathematical Monthly, Vol.128, December 2021)

Proposed by S. Stewart (Australia).

Prove

$$\int_0^\infty \left(1 - x^2 \sin^2\left(\frac{1}{x}\right)\right)^2 dx = \frac{\pi}{5}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We have that

$$\begin{aligned}
\int_0^\infty \left(1 - x^2 \sin^2\left(\frac{1}{x}\right)\right)^2 dx &= \int_0^\infty \frac{\left(t^2 - \sin^2(t)\right)^2}{t^6}\, dt \\
&= 2\int_0^\infty \frac{t^2 - \sin^2(t)}{t^4}\, dt - \int_0^\infty \frac{t^4 - \sin^4(t)}{t^6}\, dt = 2I_2 - I_4 \\
&= \frac{2}{3}\int_0^\infty \left(1 - \frac{s^2}{s^2+4}\right) ds - \frac{1}{5}\int_0^\infty \left(1 - \frac{s^4}{(s^2+4)(s^2+16)}\right) ds \\
&= \frac{8}{3}\int_0^\infty \frac{ds}{s^2+4} - \frac{4}{15}\int_0^\infty \left(\frac{16}{s^2+16} - \frac{1}{s^2+4}\right) ds \\
&= \frac{8}{3}\cdot\frac{\pi}{4} - \frac{4}{15}\left(2\pi - \frac{\pi}{4}\right) = \frac{\pi}{5}
\end{aligned}$$

where, for any even number $n \geq 2$,

$$\begin{aligned}
I_n &= \int_0^\infty \frac{t^n - \sin^n(t)}{t^{n+2}}\, dt \\
&= \frac{1}{(n+1)!}\int_0^\infty (t^n - \sin^n(t))\mathcal{L}[s^{n+1}](t)\, dt \\
&= \frac{1}{(n+1)!}\int_0^\infty s^{n+1}\left(\int_0^\infty (t^n - \sin^n(t))e^{-st}\, dt\right) ds \\
&= \frac{1}{(n+1)!}\int_0^\infty s^{n+1}(\mathcal{L}[t^n](s) - \mathcal{L}[\sin^n(t)](s))\, ds \\
&= \frac{1}{(n+1)!}\int_0^\infty s^{n+1}\left(\frac{n!}{s^{n+1}} - \frac{n!}{s(s^2+2^2)(s^2+4^2)\cdots(s^2+n^2)}\right) ds \\
&= \frac{1}{n+1}\int_0^\infty \left(1 - \frac{s^n}{(s^2+2^2)(s^2+4^2)\cdots(s^2+n^2)}\right) ds.
\end{aligned}$$

The general formula for the Laplace transform of $\sin^n(t)$ follows from the fact that for $n \geq 2$,

$$\begin{aligned}
s^2\mathcal{L}[\sin^n(t)](s) &= \mathcal{L}[D^2(\sin^n(t))](s) = \mathcal{L}[n(n-1)\sin^{n-2}(t) - n^2\sin^n(t)](s) \\
&= n(n-1)\mathcal{L}[\sin^{n-2}(t)](s) - n^2\mathcal{L}[\sin^n(t)](s)
\end{aligned}$$

which implies

$$\mathcal{L}[\sin^n(t)](s) = \frac{n(n-1)}{s^2+n^2}\,\mathcal{L}[\sin^{n-2}(t)](s).$$

$\square$