Problem 12287
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Prove

$$\sum_{n=1}^{\infty}\left(n\left(\sum_{k=n}^{\infty}\frac{1}{k^2}\right)^2-\frac{1}{n}\right)=\frac{1}{2}\left(3-\zeta(2)+3\zeta(3)\right).$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. For $N\geq 1$,

$$\begin{aligned}
2\sum_{n=1}^{N}\left(n\left(\sum_{k=n}^{\infty}\frac{1}{k^2}\right)^2-\frac{1}{n}\right)&=2\sum_{n=1}^{N}\left(n\left(\zeta(2)-H_{n-1}(2)\right)^2-\frac{1}{n}\right)\\
&=\left(N^2+N\right)\zeta^2(2)-4\zeta(2)\sum_{n=1}^{N}nH_{n-1}(2)+2\sum_{n=1}^{N}nH_{n-1}^2(2)-2H_N\\
&=\zeta(2)(4N+2\ln(N)+2\gamma+1)-4N\zeta(2)-2(\zeta(2)-1)\ln(N)\\
&\quad+3\zeta(3)-(2\gamma+2)\zeta(2)+2\gamma+3-2\ln(N)-2\gamma+o(1)\\
&=3-\zeta(2)+3\zeta(3)+o(1)
\end{aligned}$$

and the required identity is proved as soon as $N\to\infty$.

We used the following known facts,

$$\begin{aligned}
H_n&=\sum_{k=1}^{n}\frac{1}{k}=\ln(n)+\gamma+o(1)\\
H_n(2)&=\sum_{k=1}^{n}\frac{1}{k^2}=\zeta(2)-\frac{1}{n}+\frac{1}{2n^2}+o(1/n^2)\\
H_n(3)&=\sum_{k=1}^{n}\frac{1}{k^3}=\zeta(3)+o(1)\\
\sum_{k=1}^{n}\frac{H_{k-1}}{k^2}&=\zeta(3)+o(1).
\end{aligned}$$

Moreover, it can be verified by induction that

$$4\sum_{n=1}^{N}nH_{n-1}(2)=2(N(N+1)\,H_N(2)-H_N-N)=2(N^2+N)\zeta(2)-4N-2\ln(N)-2\gamma-1+o(1)$$

and

$$\begin{aligned}
2\sum_{n=1}^{N}nH_{n-1}^2(2)&=H_N(3)-2H_N(2)H_N+2\sum_{k=1}^{N}\frac{H_{k-1}}{k^2}-(2N+1)H_N(2)+2H_N+(N^2+N)H_N^2(2)\\
&=\zeta(3)-2\zeta(2)(\ln(N)+\gamma)+2\zeta(3)-(2N+1)(\zeta(2)-\frac{1}{N})+2(\ln(N)+\gamma)\\
&\quad+(N^2+N)\left(\zeta(2)-\frac{1}{N}+\frac{1}{2N}\right)^2+o(1)\\
&=(N^2+N)\zeta^2(2)-4N\zeta(2)-2(\zeta(2)-1)\ln(N)+3\zeta(3)-(2\gamma+2)\zeta(2)+2\gamma+3+o(1).
\end{aligned}$$

□