Problem 12286
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Let p be a prime number, and let m be a positive integer not divisible by p. Show that the coefficients of $(1+x+\cdots+x^{m-1})^{p-1}$ that are not divisible by p are alternately 1 and -1 modulo p. For example, $(1+x+x^2+x^3)^4 \equiv 1-x+x^4-x^6+x^8-x^{11}+x^{12} \pmod 5$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. For $0 \le n \le (m-1)(p-1)$, the coefficient of x^n of $(1+x+\cdots+x^{m-1})^{p-1}$ is

$$\begin{aligned}[x^n]\left(\sum_{k=0}^{m-1} x^k\right)^{p-1} &= [x^n]\left(\sum_{k=0}^{m-1} x^k\right)^{p} \cdot \frac{1-x}{1-x^m} = [x^n]\left(\sum_{k=0}^{m-1} x^k\right)^{p} \cdot \frac{1-x^{mp}}{1-x^m}\cdot(1-x)\\ &= [x^n]\left(\sum_{k=0}^{m-1} x^k\right)^{p} \cdot \sum_{j=0}^{p-1} x^{mj}\cdot(1-x) \equiv [x^n]\sum_{k=0}^{m-1} x^{pk}\cdot \sum_{j=0}^{p-1} x^{mj}\cdot(1-x) \pmod p\\ &\equiv [x^n]\sum_{k=0}^{m-1}\sum_{j=0}^{p-1} x^{pk+mj}\cdot(1-x) \pmod p.\end{aligned}$$

We notice that the map $(k,j)\to pk+mj$ is injective in $[0,m-1]\times[0,p-1]$ because $\gcd(p,m)=1$, hence the terms $\sum_{k=0}^{m-1}\sum_{j=0}^{p-1} x^{pk+mj}$ are all distinct. Therefore the double sum can be written as the sum of several blocks of consecutive powers $x^a+x^{a+1}+\cdots+x^{a+b}$. By multiplying each block by $1-x$ we obtain

$$(x^a+x^{a+1}+\cdots+x^{a+b})(1-x) = x^a - x^{a+b+1}$$

and it follows that, modulo p, the coefficients of $\left(\sum_{k=0}^{m-1} x^k\right)^{p-1}$ are alternately 1 and -1. □