Problem 12281
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Evaluate

$$\int_0^\infty \left(\frac{\cosh(x)}{\sinh^2(x)} - \frac{1}{x^2}\right)\ln^2(x)\,dx.$$

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Solution. We have that

$$\begin{aligned}
\int_0^\infty \left(\frac{\cosh(x)}{\sinh^2(x)} - \frac{1}{x^2}\right)\ln^2(x)\,dx &= \left[\left(\frac{1}{x} - \frac{1}{\sinh(x)}\right)\ln^2(x)\right]_0^\infty - 2\int_0^\infty \left(\frac{1}{x} - \frac{1}{\sinh(x)}\right)\frac{\ln(x)}{x}\,dx \\
&= 2\int_0^\infty \left(\frac{x}{\sinh(x)} - 1\right)\frac{\ln(x)}{x^2}\,dx = 2\int_0^\infty \left(\frac{x}{\sinh(x)} - 1\right)\mathcal{L}\left[t(1-\gamma-\ln(t))\right](x)\,dx \\
&= 2\int_0^\infty \frac{2xe^{-x} - 1 + e^{-2x}}{1-e^{-2x}}\left(\int_0^\infty t(1-\gamma-\ln(t))e^{-tx}\,dt\right)dx \\
&= 2\int_0^\infty t(1-\gamma-\ln(t))\left(\int_0^\infty \frac{2xe^{-x} - 1 + e^{-2x}}{1-e^{-2x}}e^{-tx}\,dx\right)dt \\
&= 2\lim_{N\to\infty}\int_0^\infty t(1-\gamma-\ln(t))\sum_{n=0}^{N-1}\left(\int_0^\infty \left(2xe^{-x} - 1 + e^{-2x}\right)e^{-(t+2n)x}\,dx\right)dt \\
&= 2\lim_{N\to\infty}\int_0^\infty t(1-\gamma-\ln(t))\sum_{n=0}^{N-1}\left(\frac{2}{(t+2n+1)^2} - \frac{1}{t+2n} + \frac{1}{t+2(n+1)}\right)dt \\
&= 2\lim_{N\to\infty}\sum_{n=0}^{N-1}\int_0^\infty (1-\gamma-\ln(t))\left(-\frac{2(n+1)}{(t+2n+1)^2} + \frac{2}{t+2n+1} + \frac{2n}{t+2n} - \frac{2(n+1)}{t+2(n+1)}\right)dt \\
&= 2(1-\gamma)\lim_{N\to\infty}A_N - 2\lim_{N\to\infty}B_N = 2(1-\gamma)(-\ln(2)) - 2\left(-\ln(2) + \frac{\ln(2)^2}{2} + \ln(2)\ln(\pi)\right) \\
&= \boxed{-\ln(2)(\ln(2\pi^2) - 2\gamma)}
\end{aligned}$$

where for $N \to \infty$, $\ln(N!) = N\ln(N) - N + \frac{\ln(N)}{2} + \frac{\ln(2\pi)}{2} + o(1)$,

$$\begin{aligned}
A_N &= 2\sum_{n=0}^{N-1}\left((n+1)\ln(n+1) - n\ln(n) - \ln(2n+1) + \ln(2) - 1\right) \\
&= 2N\ln(N) - 2\ln\left(\frac{(2N)!}{2^N N!}\right) + 2(\ln(2)-1)N \to -\ln(2)
\end{aligned}$$

and

$$\begin{aligned}
B_N &= \sum_{n=0}^{N-1}\big(2\ln(2)((n+1)\ln(n+1) - n\ln(n)) + (n+1)\ln^2(n+1) - n\ln^2(n) \\
&\qquad -2\ln(2n+1) - \ln^2(2n+1) + \ln^2(2)\big) \\
&= 2\ln(2)N\ln(N) + N\ln^2(N) - 2\ln\left(\frac{(2N)!}{2^N N!}\right) - \sum_{n=0}^{N-1}\ln^2(2n+1) + N\ln^2(2) \\
&= N\ln^2(N) + (2\ln(2)-2)N\ln(N) + (\ln^2(2) - 2\ln(2) + 2)N - \ln(2) - \sum_{n=0}^{N-1}\ln^2(2n+1) + o(1) \\
&\to -\ln(2) + \frac{\ln(2)^2}{2} + \ln(2)\ln(\pi).
\end{aligned}$$

Notice that by Euler-Maclaurin Formula, there exists some constant c such that

$$\sum_{n=1}^{N} \ln^2(n) = N\ln^2(N) - 2N\ln(N) + 2N + \frac{\ln^2(N)}{2} + c + o(1).$$

Therefore

$$\begin{aligned}
\sum_{n=0}^{N-1} \ln^2(2n+1) &= \sum_{n=1}^{2N} \ln^2(n) - \sum_{n=1}^{N} \ln^2(2n) \\
&= \sum_{n=1}^{2N} \ln^2(n) - \sum_{n=1}^{N} \left(\ln^2(n) - 2\ln(2)\ln(n) + \ln^2(2)\right) \\
&= \sum_{n=1}^{2N} \ln^2(n) - \sum_{n=1}^{N} \ln^2(n) + 2\ln(2)\ln(N!) - N\ln^2(2) \\
&= N\ln^2(N) + (2\ln(2) - 2)N\ln(N) + (\ln^2(2) - 2\ln(2) + 2)N - \frac{\ln(2)^2}{2} - \ln(2)\ln(\pi) + o(1).
\end{aligned}$$

□