Problem 12279
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Let j and n be positive integers of opposite parity with $j < n$. Prove

$$\sum_{k=j}^{n} \frac{(-1)^k (k-1)!}{2^k} \binom{k}{j} \left\{ {n \atop k} \right\} = 0$$

where $\left\{ {n \atop k} \right\}$ are the Stirling numbers of the second kind.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We introduce the polynomials

$$R_n(z) = \sum_{k=1}^{n} (k-1)! \left\{ {n \atop k} \right\} z^k.$$

We have that $R_1(z) = z$, and for $n \geq 2$,

$$\begin{aligned} z(z+1)\frac{d}{dz}\left(R_{n-1}(z)\right) &= (z^2+z)\sum_{k\geq 1}(k-1)!\left\{ {n-1 \atop k} \right\} k z^{k-1} = \sum_{k\geq 1} k! \left\{ {n-1 \atop k} \right\} (z^{k+1} + z^k) \\ &= \sum_{k\geq 1} k! \left\{ {n-1 \atop k} \right\} z^{k+1} + \sum_{k\geq 1} k! \left\{ {n-1 \atop k} \right\} z^k \\ &= \sum_{k\geq 1} (k-1)! \left\{ {n-1 \atop k-1} \right\} z^k + \sum_{k\geq 1} (k-1)! k \left\{ {n-1 \atop k} \right\} z^k \\ &= \sum_{k\geq 1} (k-1)! \left\{ {n \atop k} \right\} z^k = R_n(z) \end{aligned}$$

where we used $\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k\left\{ {n-1 \atop k} \right\}$ and $\left\{ {n \atop k} \right\} = 0$ for $0 < n < k$.

Going back to our problem, we notice that

$$P_n(x) := \sum_{j=0}^{n} \left(\sum_{k=j}^{n} \frac{(-1)^k (k-1)!}{2^k} \binom{k}{j} \right) x^j = \sum_{k\geq 1} (k-1)! \left\{ {n \atop k} \right\} \frac{(-1)^k}{2^k} \sum_{j=0}^{k} \binom{k}{j} x^j = R_n\left(-\frac{1+x}{2}\right).$$

Hence it suffices to show that for $n \geq 2$, P_n is an even polynomial when n is even and P_n is odd when n is odd.
Indeed, for $n \geq 2$ and $z = -(1+x)/2$,

$$\begin{aligned} \frac{1-x^2}{2} \cdot \frac{d}{dx}\left(P_{n-1}(x)\right) &= \frac{1-x^2}{2} \cdot \frac{d}{dz}\left(R_{n-1}(z)\right) \cdot \frac{dz}{dx} \\ &= -\frac{1-x^2}{4} \cdot \frac{R_n(z)}{z(z+1)} = R_n(z) = P_n(x). \end{aligned}$$

It follows that if $P_{n-1}(x)$ is even then $\frac{d}{dx}\left(P_{n-1}(x)\right)$ is odd and by the above identity, $P_n(x)$ is odd too. Similarly, if $P_{n-1}(x)$ is odd then $P_n(x)$ is even. Since $P_2(x) = (x^2-1)/4$ is even, the statement holds by induction for any $n \geq 2$. $\square$