Problem 12277
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Proposed by C. Chiser (Romania).

Let A, B, and C be three pairwise commuting 2-by-2 real matrices. Show that if at least one of the matrices $A-B$, $B-C$, and $C-A$ is invertible, then the matrix

$$A^2+B^2+C^2-AB-AC-BC$$

cannot have rank 1.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We show a more general statement: *if A, B, and C are three pairwise commuting n-by-n real matrices with $n \geq 2$ and at least one of the matrices $A-B$, $B-C$, and $C-A$ is invertible, then the matrix*

$$A^2+B^2+C^2-AB-AC-BC$$

cannot have rank $n-1$.

We may assume without loss of generality that the invertible matrix is $A-B$. Since A, B, and C are pairwise commuting, it follows that

$$\begin{aligned} M &:= A^2+B^2+C^2-AB-AC-BC \\ &= (A-B)^2+(A-B)(B-C)+(B-C)^2 \\ &= (A-B)^2(I+T+T^2) \end{aligned}$$

where $T := (A-B)^{-1}(B-C)$ (note that $(A-B)^{-1}$ and $(B-C)$ commute).

Assume that the rank of M is $n-1$. Then also the rank of $I+T+T^2$ is $n-1$ because the matrix $A-B$ is invertible, which implies that $\dim(\ker(I+T+T^2)) = n-(n-1) = 1$.

On the other hand, after letting $\omega = \frac{-1+i\sqrt{3}}{2}$, we have that

$$0 = \det(I+T+T^2) = \det(T-\omega I)\det(T-\overline{\omega} I).$$

Therefore ω or $\overline{\omega}$ are eigenvalues of T, and, since T has real coefficients and ω is not a real number, then both ω and $\overline{\omega}$ are eigenvalues of T. Suppose v_1 and $v_2 = \overline{v_1}$ are eigenvectors corresponding to to the distinct eigenvalues ω and $\overline{\omega}$ respectively. Hence v_1 and v_2 are linearly independent and

$$(I+T+T^2)v_1 = (1+\omega+\omega^2)v_1 = 0, \quad (I+T+T^2)v_2 = (1+\overline{\omega}+\overline{\omega}^2)v_2 = 0,$$

which means that $\dim(\ker(I+T+T^2)) \geq 2$. Contradiction. $\square$