Problem 12276
(American Mathematical Monthly, Vol.128, October 2021)

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Prove

$$\sum_{n=2}^{\infty}\frac{1}{n+1}\sum_{k=1}^{\lfloor n/2\rfloor}\frac{1}{2^{k-1}(k-1)!(n-2k)!}=1.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Recalling the definition of double factorial $(2k-1)!!=1\cdot 3\cdots(2k-1)=\frac{(2k)!}{2^k k!}$, we have

$$\begin{aligned}
\sum_{n=2}^{\infty}\frac{1}{n+1}\sum_{k=1}^{\lfloor n/2\rfloor}\frac{1}{2^{k-1}(k-1)!(n-2k)!}&=\sum_{n=2}^{\infty}\frac{1}{(n+1)!}\sum_{k=1}^{\infty}\binom{n}{2k}(2k)(2k-1)!!\\
&=\sum_{k=1}^{\infty}(2k)(2k-1)!!\sum_{n=2k}^{\infty}\binom{n}{2k}\frac{1}{(n+1)!}\\
&=e\sum_{k=1}^{\infty}(2k)(2k-1)!!\sum_{n=2k+1}^{\infty}\frac{(-1)^{n-1}}{n!}\\
&=e\sum_{n=3}^{\infty}\frac{(-1)^{n-1}}{n!}\sum_{k=1}^{\lfloor (n-1)/2\rfloor}(2k)(2k-1)!!\\
&=e\sum_{n=3}^{\infty}\frac{(-1)^{n-1}}{n!}\sum_{k=1}^{\lfloor (n-1)/2\rfloor}((2k+1)!!-(2k-1)!!)\\
&=e\sum_{n=3}^{\infty}\frac{(-1)^{n-1}}{n!}\left(\left(2\left\lfloor\frac{n-1}{2}\right\rfloor+1\right)!!-1\right)\\
&=e\sum_{m=1}^{\infty}\frac{(2m+1)!!}{(2m+1)!}-e\sum_{m=2}^{\infty}\frac{(2m-1)!!}{(2m)!}+e\sum_{n=3}^{\infty}\frac{(-1)^n}{n!}\\
&=e\sum_{m=1}^{\infty}\frac{1}{2^m m!}-e\sum_{m=2}^{\infty}\frac{1}{2^m m!}+e\sum_{n=3}^{\infty}\frac{(-1)^n}{n!}\\
&=\frac{e}{2}+e\left(e^{-1}-\left(1-1+\frac{1}{2}\right)\right)=1,
\end{aligned}$$

where at the third line we applied

$$\begin{aligned}
\sum_{n=2k}^{\infty}\binom{n}{2k}\frac{1}{(n+1)!}&=[x^{2k}]F(x)=e[x^{2k}]\frac{e^x}{1+x}-[x^{2k}]\frac{1}{1+x}=e\sum_{n=0}^{2k}\frac{(-1)^n}{n!}-1\\
&=e\left(e^{-1}-\sum_{n=2k+1}^{\infty}\frac{(-1)^n}{n!}\right)-1=e\sum_{n=2k+1}^{\infty}\frac{(-1)^{n-1}}{n!}
\end{aligned}$$

with

$$F(x)=\sum_{j=0}^{\infty}x^j\sum_{n=j}^{\infty}\binom{n}{j}\frac{1}{(n+1)!}=\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\sum_{j=0}^{n}\binom{n}{j}x^j=\sum_{n=0}^{\infty}\frac{(x+1)^n}{(n+1)!}=\frac{e^{x+1}-1}{x+1}.$$

□