Problem 12274
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Evaluate

$$\int_0^1 \frac{\arctan(x)}{1+x^2}\left(\ln\left(\frac{2x}{1-x^2}\right)\right)^2 dx.$$

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Solution. After letting $x = \tan(t/2)$, we have

$$\begin{aligned}
\int_0^1 \frac{\arctan(x)}{1+x^2}\left(\ln\left(\frac{2x}{1-x^2}\right)\right)^2 dx &= \frac{1}{4}\int_0^{\pi/2} t\left(\ln\left(\tan(t)\right)\right)^2\, dt \\
&= \int_0^{\pi/2} t\left(\sum_{k=1}^{\infty}\frac{\cos(2(2k-1)t)}{2k-1}\right)^2 dt \\
&= \int_0^{\pi/2} t\sum_{k=1}^{\infty}\frac{\cos^2(2(2k-1)t)}{(2k-1)^2}\, dt \\
&\quad + 2\int_0^{\pi/2} t\sum_{k=2}^{\infty}\sum_{j=1}^{k-1}\frac{\cos(2(2k-1)t)\cos(2(2j-1)t)}{(2k-1)(2j-1)}\, dt \\
&= \sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\int_0^{\pi/2} t\cos^2(2(2k-1)t)\, dt \\
&\quad + 2\sum_{k=2}^{\infty}\sum_{j=1}^{k-1}\frac{1}{(2k-1)(2j-1)}\int_0^{\pi/2} t\cos(2(2k-1)t)\cos(2(2j-1)t)\, dt \\
&= \frac{\pi^2}{16}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} = \frac{\pi^2}{16}\left(\zeta(2)-\frac{\zeta(2)}{4}\right) = \frac{3\pi^2}{64}\cdot\frac{\pi^2}{6} = \boxed{\frac{\pi^4}{128}}
\end{aligned}$$

where we applied the following Fourier series

$$\begin{aligned}
\ln\left(\tan(t)\right) &= \ln\left(\sin(t)\right) - \ln\left(\cos(t)\right) \\
&= \left(-\log(2) - \sum_{n=1}^{\infty}\frac{\cos(2nt)}{n}\right) - \left(-\log(2) - \sum_{n=1}^{\infty}(-1)^n\frac{\cos(2nt)}{n}\right) \\
&= -2\sum_{k=1}^{\infty}\frac{\cos(2(2k-1)t)}{2k-1} \qquad \text{for } t\in(0,\pi/2)
\end{aligned}$$

and, for $n, m \in \mathbb{N}$ with $n \neq m$ and of the same parity, the evaluations

$$\int_0^{\pi/2} t\cos^2(2nt)\, dt = \frac{1}{16n^2}\left[4n^2t^2 + 4nt\cos(2nt)\sin(2nt) + \cos^2(2nt)\right]_0^{\pi/2} = \frac{\pi^2}{16},$$

and,

$$\begin{aligned}
\int_0^{\pi/2} t\cos(2nt)\cos(2mt)\, dt &= \frac{1}{2}\int_0^{\pi/2} t\left(\cos(2(n-m)t) + \cos(2(n+m)t)\right)\, dt \\
&= \frac{1}{2}\int_0^{\pi/2} t\cos(2(n-m)t)\, dt + \frac{1}{2}\int_0^{\pi/2} t\cos(2(n+m)t)\, dt \\
&= \frac{1}{8(n-m)^2}\left[2(n-m)t\sin(2(n-m)t) + \cos(2(n-m)t)\right]_0^{\pi/2} \\
&\quad + \frac{1}{8(n+m)^2}\left[2(n+m)t\sin(2(n+m)t) + \cos(2(n+m)t)\right]_0^{\pi/2} = 0.
\end{aligned}$$

$\square$