Problem 12273
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Proposed by H. Ohtsuka (Japan).

For $s>1$, prove the following inequalities:

$$\sum_p \frac{1}{p^s-\frac{1}{2}} < \log(\zeta(s)), \sum_p \frac{1}{p^s} < \log\left(\frac{\zeta(s)}{\sqrt{\zeta(2s)}}\right), \sum_p \frac{1}{p^s+\frac{1}{2}} < \log\left(\frac{\zeta(s)}{\zeta(2s)}\right)$$

where all the sums are taken over prime numbers p, and $\zeta(s)=\sum_{k=1}^{\infty} 1/k^s$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. It is well known that for all $s>1$,

$$\zeta(s)=\prod_p \frac{1}{1-\frac{1}{p^s}}.$$

Therefore

$$\frac{\zeta(s)}{\sqrt{\zeta(2s)}}=\prod_p \frac{\sqrt{1-\frac{1}{p^{2s}}}}{1-\frac{1}{p^s}}=\prod_p \sqrt{\frac{1+\frac{1}{p^s}}{1-\frac{1}{p^s}}} \quad , \quad \frac{\zeta(s)}{\zeta(2s)}=\prod_p \frac{1-\frac{1}{p^{2a}}}{1-\frac{1}{p^s}}=\prod_p \left(1+\frac{1}{p^s}\right),$$

and

$$\begin{aligned}
\log(\zeta(s)) &= \sum_p \left(-\log\left(1-\frac{1}{p^s}\right)\right) \\
\log\left(\frac{\zeta(s)}{\sqrt{\zeta(2s)}}\right) &= \sum_p \frac{1}{2}\log\left(\frac{1+\frac{1}{p^s}}{1-\frac{1}{p^s}}\right) \\
\log\left(\frac{\zeta(s)}{\zeta(2s)}\right) &= \sum_p \log\left(1+\frac{1}{p^s}\right).
\end{aligned}$$

Since $0<\frac{1}{p^s}<\frac{1}{2}$ for any prime p and $s>1$, it follows that the three inequalities hold as soon as for all $x\in(0,\frac{1}{2})$,

$$\frac{1}{\frac{1}{x}-\frac{1}{2}} < -\log(1-x)\,, \quad x<\frac{1}{2}\log\left(\frac{1+x}{1-x}\right)\,, \quad \frac{1}{\frac{1}{x}+\frac{1}{2}} < \log(1+x)$$

that is

$$f_1(x)=\frac{2x}{2-x}+\log(1-x)<0\,,\ f_2(x)=2x-\log(1+x)+\log(1-x)<0\,,\ f_3(x)=\frac{2x}{2+x}-\log(1+x)<0$$

which are satisfied because $f_1(0)=f_2(0)=f_3(0)=0$ and all those functions are strictly decreasing in $[0,1)$: for all $x\in(0,1)$,

$$f_1'(x)=-\frac{x^2}{(1-x)(x-2)^2}<0\,,\ f_2'(x)=-\frac{2x^2}{1-x^2}<0\,,\ f_3'(x)=-\frac{x^2}{(1+x)(x+2)^2}<0.$$

$\square$