Problem 12272
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(a) For which integers n with $n \geq 3$ do there exist distinct positive integers $a_1, \ldots, a_n$ such that $a_i + a_{i+1}$ is a power of 2 for all $i \in \{1, \ldots, n\}$? (Here subscripts are taken modulo n, so that $a_{n+1} = a_1$.)

(b) What is the answer if the word "positive" is removed from part (a)?

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution.
• There is no integer $n \geq 3$ which satisfies (a).
Let $a_1, \ldots, a_n$ be distinct positive integers such that $a_i + a_{i+1}$ is a power of 2 for $i = 1, \ldots, n$. Without loss of generality, we may assume that $a_1 = \max_{1 \leq i \leq n} a_i$. Since $a_1 + a_2 = 2^t$ and $a_n + a_1 = 2^s$ for some integers $t, s \geq 0$ then

$$a_1 > a_2 = 2^t - a_1 > 0 \implies 2^{t-1} < a_1 < 2^t, \quad \text{and} \quad a_1 > a_n = 2^s - a_1 > 0 \implies 2^{s-1} < a_1 < 2^s.$$

Since a_1 is between a unique pair of consecutive powers of 2, it follows that $t = s$ which implies that $a_2 = a_n$. Contradiction.

• Any integer $n \geq 3$ such that $n \neq 4$ satisfies (b).
If n is odd, then the system of equations

$$\begin{cases} a_i + a_{i+1} = 2^i & \text{for } i = 1, \ldots, n-1 \\ a_n + a_1 = 2^n \end{cases}$$

has a unique solution $\mathbf{a}_n = (a_1, a_2, \ldots, a_n)$ where

$$a_i = \frac{2^i + (-1)^i(1 - 2^n)}{3} \qquad \text{for } i = 1, \ldots, n$$

are distinct integers (the numerator is divisible by 3 when n is odd). For example

$$\mathbf{a}_3 = (3, -1, 5), \quad \mathbf{a}_5 = (11, -9, 13, -5, 21).$$

If $n = 4$ then there are no distinct integers a_1, a_2, a_3, a_4 such that

$$a_1 + a_2 = 2^A,\ a_2 + a_3 = 2^B,\ a_3 + a_4 = 2^C,\ a_4 + a_1 = 2^D,$$

for some non-negative integers A, B, C, D. Otherwise $a_1 + a_2 + a_3 + a_4 = 2^A + 2^C = 2^B + 2^D$ and if $A = \max\{A, B, C, D\}$ (the other cases are similar) then $A > B$ (because $a_1 \neq a_3$), and $A > D$ (because $a_2 \neq a_4$), and we find a contradiction:

$$2^A = 2^{A-1} + 2^{A-1} \geq 2^B + 2^D = 2^A + 2^C > 2^A.$$

If $n = 2m$ is even with $m \geq 3$ then the system of equations

$$\begin{cases} a_1 + a_2 = 2^{m-1} \\ a_3 + a_4 = 2^m \\ a_{2j+3} + a_{2j+4} = 2^j & \text{for } j = 1, \ldots, m-2 \\ a_{2j} + a_{2j+1} = 2^j & \text{for } j = 1, \ldots, m-1 \\ a_{2m} + a_1 = 2^m. \end{cases}$$

has a unique solution such that $a_1 = 0$ where $a_1, a_2, \ldots, a_n$ are distinct integers.
Note that we may write this solution recursively: let $\mathbf{a}_6 = (0, 4, -2, 10, -6, 8)$ then for $m \geq 3$, if $\mathbf{a}_{2m} = (a_1, a_2, \ldots, a_{2m})$ then

$$\mathbf{a}_{2m+2} = (2a_1, 2a_2, 2 - 2a_2, 2 + 2a_4, 2 - 2a_4, 2a_4, \ldots, 2a_m)$$

where

$$(2 - 2a_2) + (2 + 2a_4) = 2(2 - (a_2 + a_3) + (a_3 + a_4) = 2(2 - 2 + 2^m) = 2^{m+1}.$$

□