Problem 12270
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Let $a_0 = 1$, and let $a_{n+1} = a_n + e^{-a_n}$ for $n \geq 0$. Show that the sequence whose nth term is $e^{a_n} - n - \frac{\ln(n)}{2}$ converges.

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Solution. The sequence $(a_n)_n$ is positive and increasing, therefore it has a limit in $\mathbb{R}^+ \cup \{+\infty\}$. If the limit is $l \in \mathbb{R}^+$ then, from the recurrence, we find $l = l + e^{-l}$, that is $e^{-l} = 0$ which is impossible. Hence $\lim_{n\to\infty} a_n = +\infty$.
Let $x_n = a_{n+1} - a_n = e^{-a_n}$, then $\lim_{n\to\infty} x_n = 0^+$ and $x_{n+1} = e^{-a_{n+1}} = e^{-a_n - x_n} = x_n e^{-x_n}$. Therefore

$$\frac{1}{x_{n+1}} - \frac{1}{x_n} = \frac{e^{x_n} - 1}{x_n} = 1 + \frac{x_n}{2} + \frac{x_n^2}{6} + O(x_n^3). \tag{1}$$

Note that by (1) and by Stolz-Cesaro theorem,

$$\lim_{n\to\infty} \frac{1}{nx_n} = \lim_{n\to\infty} \frac{\frac{1}{x_n}}{n} = \lim_{n\to\infty} \frac{\frac{1}{x_{n+1}} - \frac{1}{x_n}}{n+1-n} = 1$$

and

$$\lim_{n\to\infty} \frac{\frac{1}{x_n} - n}{\ln(n)} = \lim_{n\to\infty} \frac{\frac{1}{x_{n+1}} - \frac{1}{x_n} - 1}{\ln(n+1) - \ln(n)} = \lim_{n\to\infty} \frac{\frac{x_n}{2} + O(x_n^2)}{\frac{1}{n} + O(1/n^2)} = \lim_{n\to\infty} \frac{nx_n}{2} \frac{1 + O(x_n)}{1 + O(1/n)} = \frac{1}{2}.$$

It follows that $\frac{1}{x_n} = n + O(\ln(n))$ and

$$a_n = \ln\left(\frac{1}{x_n}\right) = \ln(n + O(\ln(n))) = \ln(n) + \ln\left(1 + \frac{O(\ln(n))}{n}\right) = \ln(n) + O\left(\frac{\ln(n)}{n}\right). \tag{2}$$

Moreover, again by (1),

$$\lim_{n\to\infty} n^2\left(\frac{1}{x_{n+1}} - \frac{1}{x_n} - 1 - \frac{x_n}{2}\right) = \lim_{n\to\infty} n^2\left(\frac{x_n^2}{6} + O(x_n^3)\right) = \lim_{n\to\infty} \frac{(nx_n)^2}{6}(1 + O(x_n)) = \frac{1}{6}$$

which implies

$$b_n := \frac{1}{x_{n+1}} - \frac{1}{x_n} - 1 - \frac{x_n}{2} = O\left(\frac{1}{n^2}\right). \tag{3}$$

Hence, by (2) and (3),

$$\begin{aligned}
e^{a_n} &= \frac{1}{x_n} = \frac{1}{x_0} + \sum_{k=0}^{n-1}\left(\frac{1}{x_{k+1}} - \frac{1}{x_k}\right) = e + \sum_{k=0}^{n-1}\left(1 + \frac{x_k}{2} + b_k\right) \\
&= e + n + \frac{1}{2}\sum_{k=0}^{n-1} x_k + \sum_{k=0}^{n-1} b_k = e + n + \frac{1}{2}\sum_{k=0}^{n-1}(a_{k+1} - a_k) + \sum_{k=0}^{n-1} b_k \\
&= e + n + \frac{a_n}{2} - \frac{1}{2} + \sum_{k=0}^{n-1} b_k = n + \frac{\ln(n)}{2} + e - \frac{1}{2} + \sum_{k=0}^{n-1} b_k + O\left(\frac{\ln(n)}{n}\right).
\end{aligned}$$

Finally we may conclude that the required sequence is convergent:

$$\lim_{n\to\infty}\left(e^{a_n} - n - \frac{\ln(n)}{2}\right) = e - \frac{1}{2} + \sum_{k=0}^{\infty} b_k$$

where the series on the right is absolutely convergent because by $b_k = O\left(\frac{1}{k^2}\right)$. □