Problem 12269
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Let ABC be an acute triangle. Suppose that D, E, and F are points on sides BC, CA, and AB, respectively, such that FD is perpendicular to BC, DE is perpendicular to CA, and EF is perpendicular to AB. Prove

$$\frac{AF}{AB} + \frac{BD}{BC} + \frac{CE}{CA} = 1.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let $a = BC$, $b = CA$, $c = AB$, $x = BD$, $y = CE$, and $z = AF$. By the condition of perpendicularity:

$$\frac{z}{b-y} = \frac{AF}{AE} = \cos(A) = \frac{b^2+c^2-a^2}{2bc} \implies \frac{z}{c} = \frac{b^2+c^2-a^2}{2c^2}\left(1-\frac{y}{b}\right).$$

Similarly we get

$$\frac{x}{a} = \frac{c^2+a^2-b^2}{2a^2}\left(1-\frac{z}{c}\right) \quad \text{and} \quad \frac{y}{b} = \frac{a^2+b^2-c^2}{2b^2}\left(1-\frac{x}{a}\right).$$

By solving the linear system of these three equations with respect to x/a, y/b, z/c we find

$$\frac{x}{a} = \frac{a^2+c^2-b^2}{a^2+b^2+c^2}, \quad \frac{y}{b} = \frac{a^2+b^2-c^2}{a^2+b^2+c^2}, \quad \frac{z}{c} = \frac{b^2+c^2-a^2}{a^2+b^2+c^2}.$$

Therefore

$$\frac{AF}{AB} + \frac{BD}{BC} + \frac{CE}{CA} = \frac{x}{a} + \frac{y}{b} + \frac{x}{c} = \frac{a^2+c^2-b^2+a^2+b^2-c^2+b^2+c^2-a^2}{a^2+b^2+c^2} = 1.$$

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