Problem 12267
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Proposed by M. Bataille (France).

Let x, y, and z be nonnegative real numbers such that $x+y+z=1$. Prove

$$(1-x)\sqrt{x(1-y)(1-z)}+(1-y)\sqrt{y(1-z)(1-x)}+(1-z)\sqrt{z(1-x)(1-y)}\geq 4\sqrt{xyz}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Since

$$\begin{aligned}\sqrt{(1-y)(1-z)}=\sqrt{(x+z)(x+y)}&=\sqrt{x^2+x(y+z)+yz}\\&\geq\sqrt{x^2+2x\sqrt{yz}+yz}=x+\sqrt{yz},\end{aligned}$$

and, by Muirhead's inequality,

$$\begin{aligned}\sqrt{x^3}(y+z)+\sqrt{y^3}(z+x)+\sqrt{z^3}(x+y)&=\sum_{\text{sym}}x^{3/2}y^1z^0\\&\geq\sum_{\text{sym}}x^{3/2}y^{1/2}z^{1/2}=2\sqrt{xyz}(x+y+z),\end{aligned}$$

it follows that

$$\begin{aligned}&(1-x)\sqrt{x(1-y)(1-z)}+(1-y)\sqrt{y(1-z)(1-x)}+(1-z)\sqrt{z(1-x)(1-y)}\\&\quad\geq(y+z)\sqrt{x}(x+\sqrt{yz})+(z+x)\sqrt{y}(y+\sqrt{zx})+(x+y)\sqrt{z}(z+\sqrt{xy})\\&\quad=\sqrt{x^3}(y+z)+\sqrt{y^3}(z+x)+\sqrt{z^3}(x+y)+2\sqrt{xyz}(x+y+z)\\&\quad\geq 4\sqrt{xyz}(x+y+z)=4\sqrt{xyz}.\end{aligned}$$

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