Problem 12265
(American Mathematical Monthly, Vol.128, August-September 2021)

Proposed by R. Dempsey (USA).

For a fixed positive integer k, let $a_0 = a_1 = 1$ and $a_n = a_{n-1} + (k-n)^2 a_{n-2}$ for $n \geq 2$. Show that $a_k = (k-1)!$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma “Tor Vergata”, via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. The property holds for $k = 1$: $a_1 = 1 = (1-1)!$. For $k = 2$, we have

$$a_2 = 1 + (k-2)^2 \cdot 1 = 1 = (2-1)!.$$

Let $k \geq 3$, and for $0 \leq n \leq k-2$, let

$$b_n = \sum_{j=0}^{n} (-1)^{n-j} \prod_{i=0, i\neq j}^{n} (k-1-i).$$

Then

$$b_0 = 1, \qquad b_1 = -(k-2) + (k-1) = 1.$$

Moreover, for $2 \leq n \leq k-2$,

$$\begin{aligned}
b_n &= -(k-1-n) b_{n-1} + \prod_{i=0}^{n-1} (k-1-i) \\
&= b_{n-1} - (k-n) \left(b_{n-1} - \prod_{i=0}^{n-2} (k-1-i) \right) \\
&= b_{n-1} - (k-n) \left(-(k-n) b_{n-2} + \prod_{i=0}^{n-2} (k-1-i) - \prod_{i=0}^{n-2} (k-1-i) \right) \\
&= b_{n-1} + (k-n)^2 b_{n-2}.
\end{aligned}$$

Hence, satisfying the same recurrence relation, it follows that $a_n = b_n$ for all $0 \leq n \leq k-2$. Finally we compute a_k,

$$\begin{aligned}
a_k &= a_{k-1} + (k-k)^2 a_{k-2} = a_{k-1} \\
&= a_{k-2} + (k-(k-1))^2 a_{k-3} = a_{k-2} + a_{k-3} \\
&= \sum_{j=0}^{k-2} (-1)^{k-2-j} \prod_{i=0, i\neq j}^{k-2} (k-1-i) + \sum_{j=0}^{k-3} (-1)^{k-3-j} \prod_{i=0, i\neq j}^{k-3} (k-1-i) \\
&= (k-1)! \sum_{j=0}^{k-2} \frac{(-1)^{k-j}}{k-1-j} - (k-1)! \sum_{j=0}^{k-3} \frac{(-1)^{k-j}}{k-1-j} = (k-1)!.
\end{aligned}$$

□