Problem 12262
(American Mathematical Monthly, Vol.128, June-July 2021)

Proposed by Li Zhou (USA).

For a nonnegative integer m, let

$$A_m = \sum_{k=0}^{\infty} \left(\frac{1}{(6k+1)^{2m+1}} - \frac{1}{(6k+5)^{2m+1}} \right).$$

Prove $A_0 = \pi\sqrt{3}/6$ and, for $m \geq 1$,

$$2A_m + \sum_{n=1}^{m} \frac{(-1)^n \pi^{2n}}{(2n)!} A_{m-n} = \frac{(-1)^m (4^m+1)\sqrt{3}}{2(2m)!} \left(\frac{\pi}{3}\right)^{2m+1}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. From the series representation of the digamma function $\psi(z) = -\gamma + \sum_{k=0}^{\infty} \left(\frac{1}{k+1} - \frac{1}{k+z}\right)$, it follows that

$$A_0 = \sum_{k=0}^{\infty} \left(\frac{1}{6k+1} - \frac{1}{6k+5} \right) = \frac{\psi(5/6) - \psi(1/6)}{6} = \frac{\pi \cot(\pi/6)}{6} = \frac{\pi\sqrt{3}}{6}$$

where we applied the reflection formula $\psi(1-z) - \psi(z) = \pi \cot(\pi z)$.
As regards the identity, we multiply both sides by z^{2m} and we take the sum over all $m \geq 0$. We are going to verify that both sides lead the same function of z and therefore the equality holds.
For the right-hand side:

$$\begin{aligned}
\text{RHS}(z) &= \sum_{m=0}^{\infty} \frac{(-1)^m (4^m+1)\sqrt{3}}{2(2m)!} \left(\frac{\pi}{3}\right)^{2m+1} z^{2m} = \frac{\sqrt{3}\pi}{6} \left(\cos\left(\frac{4\pi z}{3}\right) + \cos\left(\frac{\pi z}{3}\right) \right) \\
&= \frac{\pi}{\sqrt{3}} \cos\left(\frac{\pi z}{2}\right) \cos\left(\frac{\pi z}{6}\right).
\end{aligned}$$

For the left-hand side:

$$\begin{aligned}
\text{LHS}(z) &= \sum_{m=0}^{\infty} A_m z^{2m} + \sum_{m=0}^{\infty} \sum_{n=0}^{m} \frac{(-1)^{m-n} \pi^{2(m-n)}}{(2(m-n))!} A_n z^{2m} \\
&= \sum_{m=0}^{\infty} A_m z^{2m} + \sum_{n=0}^{\infty} A_n z^{2n} \sum_{m=n}^{\infty} \frac{(-1)^{m-n} \pi^{2(m-n)}}{(2(m-n))!} z^{2(m-n)} \\
&= \sum_{m=0}^{\infty} A_m z^{2m} + \sum_{n=0}^{\infty} A_n z^{2n} \cos(\pi z) = (1+\cos(\pi z)) \sum_{n=0}^{\infty} A_n z^{2n} \\
&= (1+\cos(\pi z)) \sum_{k=0}^{\infty} \left(\sum_{n=0}^{\infty} \frac{z^{2n}}{(6k+1)^{2n+1}} - \sum_{n=0}^{\infty} \frac{z^{2n}}{(6k+5)^{2n+1}} \right) \\
&= (1+\cos(\pi z)) \sum_{k=0}^{\infty} \left(\frac{6k+1}{(6k+1)^2 - z^2} - \frac{6k+5}{(6k+5)^2 - z^2} \right) \\
&= \frac{1+\cos(\pi z)}{2} \sum_{k=0}^{\infty} \left(\frac{1}{6k+1+z} - \frac{1}{6k+5+z} + \frac{1}{6k+1-z} - \frac{1}{6k+5-z} \right) \\
&= \frac{1+\cos(\pi z)}{12} \left(\psi((5+z)/6) - \psi((1-z)/6) + \psi((5-z)/6) - \psi((1+z)/6) \right) \\
&= \frac{\pi(1+\cos(\pi z))}{12} \left(\cot\left(\frac{\pi(1-z)}{6}\right) + \cot\left(\frac{\pi(1+z)}{6}\right) \right) \\
&= \frac{\pi}{\sqrt{3}} \cos\left(\frac{\pi z}{2}\right) \cos\left(\frac{\pi z}{6}\right)
\end{aligned}$$

where at the end we applied again the reflection formula. □