Problem 12261
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Proposed by A. Stadler (Switzerland).

Let a_n *be the number of equilateral triangles whose vertices are chosen from the vertices of the* n*-dimensional cube. Compute* $\lim_{n\to\infty} \frac{na_n}{8^n}$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \{0,1\}^n$ be vertices of n-cube and let

$$S_1 = \{i \in [1,n] \;:\; (\mathbf{v}_2)_i \neq (\mathbf{v}_3)_i\},$$
$$S_2 = \{i \in [1,n] \;:\; (\mathbf{v}_1)_i \neq (\mathbf{v}_3)_i\},$$
$$S_3 = \{i \in [1,n] \;:\; (\mathbf{v}_1)_i \neq (\mathbf{v}_2)_i\}.$$

Note that $S_1 = (S_2 \setminus S_3) \cup (S_3 \setminus S_2)$ and therefore $|S_1| = |S_2| + |S_3| - 2|S_2 \cap S_3|$.
The triangle with vertices $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is equilateral if and only if $|S_1| = |S_2| = |S_3| > 0$, that is

$$|S_2| = |S_3| = 2|S_2 \cap S_3| = 2k$$

for some positive integer k and its side length is $\sqrt{2k}$.
In order to count such triangles, we first choose the vertex $\mathbf{v}_1$ in 2^n ways, then we select the vertex $\mathbf{v}_2$ by choosing the set of indices S_3 in $\binom{n}{2k}$ ways. Finally the vertex $\mathbf{v}_3$ is determined as soon as we choose the set $S_2 \cap S_3$ in $\binom{2k}{k}$ ways, and the set $S_2 \setminus S_3$ in $\binom{n-2k}{k}$ ways. Since each triple $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ has been counted 3! times, we divide the final product by 6. Therefore the number of equilateral triangles of side length $\sqrt{2k}$ is

$$\frac{2^n}{3!}\binom{n}{2k}\binom{2k}{k}\binom{n-2k}{k} = \frac{2^n}{6}\binom{n}{3k}\frac{(3k)!}{(k!)^3},$$

and by summing over all $1 \leq k \leq n/3$ we find the total number of equilateral triangles in the n-cube:

$$a_n = \frac{2^n}{6}\sum_{k=1}^{n/3}\binom{n}{3k}\frac{3k!}{(k!)^3}.$$

Starting from $n = 3$, the first terms of the sequence are: $8, 64, 320, 2240, 17920, 121856, 831488$.
By using the Stirling's approximation we get

$$a_n = \frac{2^n}{6}\sum_{k=1}^{n/3}\binom{n}{3k}\frac{3k!}{(k!)^3} \sim \frac{2^n}{6}\sum_{k=1}^{n/3}\binom{n}{3k}\frac{(6\pi k)^{1/2}\,(3k)^{3k}e^{-3k}}{(2\pi k)^{3/2}\,k^{3k}e^{-3k}} = \frac{2^n}{4\sqrt{3}\pi}\sum_{k=1}^{n/3}\binom{n}{3k}\frac{3^{3k}}{k}$$
$$\sim \frac{2^n}{4\sqrt{3}\pi n}\sum_{k=1}^{n/3}\binom{n+1}{3k+1}3^{3k+1} \sim \frac{2^n}{4\sqrt{3}\pi n}\cdot\frac{(1+3)^{n+1}}{3} = \frac{8^n}{3\sqrt{3}\pi n}.$$

Finally

$$\lim_{n\to\infty}\frac{na_n}{8^n} = \frac{1}{3\sqrt{3}\pi}.$$

□